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Devoted to All Branches of Optics

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SEPTEMBER, 1920

A Focal-Length Equation 243
C. W. WOODWORTH

Note on Professor Woodworth's Formula for Focal Length 245
JAMES P. C. SOUTHALL

Some General Characteristics of Spectro-photo-electrical Conduction in Solids 249
W. W. COBLENTZ

On the Relation Between Photographic Density, Light Intensity, and Exposure Time 255
FRANK E. ROSS

The Cemented Telescope Objective of Barium Crown and Flint 274
I. C. GARDNER

Note on the Path of a Ray Through a Prism in a Principal Section 283
JAMES P. C. SOUTHALL

The Treatment of Certain Special Cases in Lens Calculation 286
C. W. WOODWORTH

Note on Calculation of Path of Ray Through a Symmetrical Optical Instrument 294
JAMES P. C. SOUTHALL

A New Method for the Measurement of Photographic Filter Factors 300
RAYMOND DAVIS

1919 Report of Standards Committee on Pyrometry 305
W. E. FORSYTHE, Chairman

A Coöperative College Course in Illuminating Engineering 337
F. E. CADY

The Absorption of Heat in Glass 340
A. Q. TOOL AND C. G. EICHLIN

Resonance Potentials and Low-Voltage Arcs for Metals of the Second Group of the Periodic Table 364
F. L. MOHLER, PAUL D. FOOTE AND W. F. MEGGERS

The Precision of Photometric Measurements 371
F. K. RICHTMYER AND E. C. CRITTENDEN

Preliminary Note on the Relations Between the Quality of Color and the Spectral Distribution of Light in the Stimulus 388
IRWIN G. PRIEST

Note on the Relation Between the Frequencies of Complementary Hues . . 402
IRWIN G. PRIEST

Reviews 405

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A FOCAL-LENGTH EQUATION.

BY

C. W. WOODWORTH.

No general equation for calculating the focal length of a lens system appears to have been published. The mathematical equivalent of the author's graphic method of calculating paraxial ray courses * may be stated in the form of such an equation as follows:

$$j_m = j_{m-1} + i_m\left(1 - \sum_{k=1}^{k=m-1} \frac{j_k d_k}{n_{k+1}}\right)$$

in which

i = surface constant $= \frac{n'-n}{r} =$ n/ focal length;

j = lens constant $=$ n/ focal length;

d = separation between the surfaces; and

n = index of refraction.

The focal length of a lens system, stated in another way, is equal to the sum of the lens constants plus the sum of the correction factors for the successive separations.

The following is a calculation of the telescopic objective given in Southall's "Principles and Methods of Geometrical Optics," pp. 319–321:

					$\Sigma \frac{jd}{n}$
	i_1	.0086053512	r_1 =	59.8	
i_2	Σ	−.0000324321	d_1 =	1.	.0056815999
	i_2	.0057082640	r_2 =	−90.15	
	j_2	.0142811831	d_2 =	.013	.0001856554
i_3	Σ	.0000424008	n″ =	1.6121	.0058672553
	i_3	−.0072266824	r_3 =	−84.7	
	j_3	.0070969015	d_3 =	1.	.0044022713
i_4	Σ	.0000153317			.0102695266
	i_4	−.0014929268	r_4 =	410.	
	j_4	.0056193064			

* *Proc. Calif. Acad. Sci.*, Vol. i, p. 192.

The procedure is (1) arrange the data as in column two; (2) determine the lens constant for each value of r; (3) determine first number in column three $= \frac{j_1 d_1}{n_2}$; (4) determine first correction by multiplying the previous number by i_2; (5) determine lens constant of the first lens by addition; (6) determine the second number in column three $= \frac{j_2 d_2}{n_3}$; (7) determine second correction by adding the two numbers in column three and multiplying by i_3; (8) determine constant for first three surfaces by adding the constants of the three surfaces and the corrections for the first two separations; (9) determine the number corresponding with the third separation $= \frac{j_3 d_3}{n_4}$; (10) determine the third correction by multiplying the sum of the three numbers of column three to i_4; and, finally, (11) determine the constant for the whole system by adding the four surface constants and the three correction numbers.

Because of the alternate addition and multiplication, machine calculation will be very much more rapid than the use of logarithms.

The focal length of the lens is $n(=1)/_{j4} = 177.9579060$ the distance from the last lens vertex to the principal plane, is the sum of the corrections divided by the lens constant $\Sigma/j_4 = 1.8275835$ and the difference between these numbers the focal distance from the last lens vertex $= 176.1303625$.

The same lens calculated in the opposite direction gives different correction values corresponding with the different relative location of the other principal plane with respect to the plane of the other outer lens surface, but gives, of course, precisely the same final value to the lens constant of the system and its focal length.

The use of this equation accomplishes a considerable saving in the time required for a calculation, since it eliminates some of the non-essential steps in the ordinary methods.

UNIVERSITY OF CALIFORNIA.

NOTE ON PROFESSOR WOODWORTH'S FORMULA FOR FOCAL LENGTH.

BY

JAMES P. C. SOUTHALL.

THE formula for the refracting power of a centered optical system given by Professor Woodworth in his paper, entitled "A Focal-length Equation," which is published in this number of THE JOURNAL OF THE OPTICAL SOCIETY,* is extremely compact and convenient and lends itself readily to numerical computations, as he shows. So far as I am aware, this formula has never been published before in precisely this form, although it is difficult to see how it could have escaped notice. It may be derived in various ways, but perhaps most easily by using the ordinary formulæ for the combination of two optical systems, as given, for example, in Chapter XI of my text-book on "Mirrors, Prisms and Lenses." Using a notation similar to that employed in that book (and in conformity with the notation recommended in the first issue of THE JOURNAL OF THE OPTICAL SOCIETY), let

$$F_k = (n_{k+1} - n_k) R_k$$

denote the refracting power of the kth surface whose curvature is denoted by R_k and which separates the two media of indices n_k and n_{k+1}. Moreover, let $F_{a,b}$ denote the refracting power of the entire system of spherical refracting surfaces which is comprised between the media of indices n_a and n_b where the subscripts a, b denote here two integers; concerning which it is agreed that in case $a > b$, we must put $F_{a,b} = 0$, and if $a = b$ then $F_{a,a} = F_a$. Let the vertex of the kth surface be designated by A_k, and put $d_k = A_k A_{k+1}$ and $c_k = d_k / n_{k+1}$ where c denotes therefore the reduced thickness of the medium as measured along the axis of the system. And, finally, let $H'_{1,k}$ designate the second principal point of the system composed of the first k surfaces whose refracting power is equal to $F_{1,k}$; and put

$$n_{k+1} x_k = H'_{1,k} A_{k+1}.$$

Then evidently we can write the following recurrent formula:

$$F_{1,k} = F_{1,k-1} + F_k (1 - x_{k-1} F_{1,k-1}). \qquad (1)$$

* Preceding article.

It is easy to see that

$$x_k \cdot F_{1,k} = x_{k-1} \cdot F_{1,k-1} + c_k \cdot F_{1,k},$$

and therefore that

$$x_{k-1} \cdot F_{1,k-1} = \sum_{p=1}^{p=k-1} c_p \cdot F_{1,p}; \tag{2}$$

which in combination with (1) above gives immediately Professor Woodworth's equation expressed, however, in terms of my symbols, namely:

$$F_{1,k} = F_{1,k-1} + F_k \left\{ 1 - \sum_{p=1}^{p=k-1} c_p \cdot F_{1,p} \right\}, \tag{3}$$

where k must be given in succession all integral values from $k=2$ to $k=k$. Professor Woodworth's symbols j_k and i_k are equivalent to $F_{1,k}$ and F_k as here used.

It is surprising that a relation which is at once so convenient and obvious seems never to have been given in any of the books. Practically it is equivalent to two formulæ published by T. Smith in a paper entitled "Note on Optical Imagery" in *The Proceedings of the Physical Society of London*, xxv (1912–1913), pp. 239–244. Thus, if equation (1) is differentiated with respect to F_k, the partial derivative of $F_{1,k}$ will be found to be the term contained within the brackets; that is,

$$\frac{\partial F_{1,k}}{\partial F_k} = 1 - x_{k-1} \cdot F_{1,k-1}; \tag{4}$$

and hence formula (1) may be written:

$$F_{1,k} = F_{1,k-1} + F_k \frac{\partial F_{1,k}}{\partial F_k}. \tag{5}$$

And, moreover, since

$$1 - x_{k-1} \cdot F_{1,k-1} = 1 - x_{k-2} \cdot F_{1,k-2} - c_{k-1} \cdot F_{1,k-1},$$

we may write also:

$$\frac{\partial F_{1,k}}{\partial F_k} = \frac{\partial F_{1,k-1}}{\partial F_{k-1}} - c_{k-1} \cdot F_{1,k-1}. \tag{6}$$

Equations (5) and (6) are the formulæ of T. Smith referred to above. Comparing (5) with (3), we see that

$$\frac{\partial F_{1,k}}{\partial F_k} = 1 - \sum_{p=1}^{p=k-1} c_p \cdot F_{1,p}, \tag{7}$$

and equation (6) follows at once from (7). This shows how (5) and (6) are together equivalent to (3).

If the total number of surfaces is denoted by m, then we may write also:

$$\left.\begin{aligned} F_{k,m} &= F_{k+1,m} + F_k \frac{\partial F_{k,m}}{\partial F_k} \\ \frac{\partial F_{k,m}}{\partial F_k} &= \frac{\partial F_{k+1,m}}{\partial F_{k+1}} - c_k . F_{k+1,m}. \end{aligned}\right\} \tag{8}$$

These latter formulæ may be obtained from (5) and (6) by imagining the light to traverse the system from the other end, so that the light is incident first on the mth surface. All that is necessary is to change the order of the double subscripts, and write subscript m in place of subscript 1 and subscript $(k+1)$ in place of subscript $(k-1)$, leaving other subscripts and everything else unchanged.

If the focal points of the system are designated by F, F', it is easy to show also that

$$A_1F = -\frac{1}{F_{1,m}} \cdot \frac{\partial F_{1,m}}{\partial F_1}, \quad A_mF' = \frac{1}{F_{1,m}} \cdot \frac{\partial F_{1,m}}{\partial F_m}. \tag{9}$$

By way of illustration of the application of formulæ (5) and (6), a calculation is given below of the famous portrait objective of Petzval with data given by von Rohr as follows:

(Distances expressed in mm.)

Radii:

$$r_1 = +52.9;\ r_2 = -41.4;\ r_3 = +436.2;\ r_4 = +194.8;\ r_5 = +36.8;$$
$$r_6 = +45.5;\ r_7 = -149.5.$$

Thicknesses:

$$d_1 = +5.8;\ d_2 = +1.5;\ d_3 = +46.6;\ d_4 = +2.2;\ d_5 = +0.7;\ d_6 = +3.6.$$

Indices of Refraction (n_D)*:*

$$n_1 = n_4 = n_6 = n_8 = 1 \text{ (air)};\ n_2 = n_7 = 1.517;\ n_3 = n_5 = 1.575.$$

The values of F_k and c_k must be calculated first, and then the work may be arranged in tabular form as follows, according to T. Smith's scheme:

	$\frac{\partial F_{1,k}}{\partial F_k}$	$F_k \cdot \frac{\partial F_{1,k}}{\partial F_k}$	$F_{1,k}$	$-c_k \cdot F_{1,k}$
k=1	+1.0000000	+0.0097732	+0.0097732	−0.0373661
k=2	+0.9626339	−0.0013486	+0.0084245	−0.0080234
k=3	+0.9546105	−0.0012584	+0.0071661	−0.3339430
k=4	+0.6206675	+0.0034054	+0.0105716	−0.0147666
k=5	+0.6059009	−0.0094672	+0.0011044	−0.0007730
k=6	+0.6051279	+0.0068759	+0.0079802	−0.0189379
k=7	+0.5861900	+0.0020272	+0.0100074	

Working through the system from the other end, we obtain also:

	$\frac{\partial F_{\kappa,7}}{\partial F_\kappa}$	$F_\kappa \cdot \frac{\partial F_{\kappa,7}}{\partial F_\kappa}$	$F_{\kappa,7}$	$-c_{\kappa-1} \cdot F_{\kappa,7}$
k=7	+1.0000000	+0.0034582	+0.0034582	−0.0082067
k=6	+0.9917933	+0.0112694	+0.0147276	−0.0103093
k=5	+0.9814840	−0.0153357	−0.0006081	+0.0008494
k=4	+0.9823334	+0.0053897	+0.0047816	−0.2228230
k=3	+0.7595105	−0.0010012	+0.0037804	−0.0036004
k=2	+0.7559101	−0.0010590	+0.0027214	−0.0104049
k=1	+0.7455052	+0.0072859	+0.0100074	

The two results are in complete agreement. We find for the focal length: $f = n_1/F_{1,7} = +99.927$ mm. or 100 mm. Moreover, the positions of the focal points are:

$$A_1F = -(1/F_{1,7})\,(\partial F_{1,7}/\partial F_1) = -74.496 \text{ mm.};$$
$$A_7F' = (1/F_{1,7})\,(\partial F_{1,7}/\partial F_7) = +58.576 \text{ mm.}$$

DEPARTMENT OF PHYSICS,
COLUMBIA UNIVERSITY,
July 1, 1920.

SOME GENERAL CHARACTERISTICS OF SPECTRO-PHOTO-ELECTRICAL CONDUCTION IN SOLIDS.*

BY

W. W. COBLENTZ.

I. INTRODUCTION.

A SUBSTANCE which changes in electrical conductivity when exposed to light is said to be photo-electrically sensitive. It is found that most of these photosensitive substances have a low electrical conductivity when they are kept in the dark. Hence, when connected in a series with a galvanometer and an electric battery, a small "dark" current flows through the circuit. On exposure to light, the photosensitive substance usually increases greatly in electrical conductivity. The reaction, or response, is said to be "photopositive." The most commonly known substance showing this type of reaction is selenium.

On a few occasions experimenters have observed that, on closing the circuit, the galvanometer deflection was less (*i.e.*, the resistance was higher) when the substance was exposed to light than when it was kept in the dark. Or, on a closed circuit, the galvanometer deflection moved in a negative direction from its zero position, when the substance was exposed to light, thus indicating an apparent rise in resistance which was higher than that which obtains in the dark. This type of response is said to be "photonegative," and is of such rare occurrence that no attempt is made to generalize the phenomenon.

Although much work has been done on the photo-electrical properties of selenium, not until within the past four years was a systematic search made to find solid substances which exhibit photo-electrical properties.[1]

In the meantime a number of spectro-photo-electrical investigations, made principally at this Bureau, on various substances, have produced results which can be used to formulate a few general characteristics of spectro-electrical sensitivity in solids.

* Published by permission of Director, Bureau of Standards.

[1] Case, *Phys. Rev.* (2), 9, p. 305; April 1917. See also papers by the writer *Proc. Phil. Soc. Wash.*, Feb 1917; *Jour. Wash. Acad. Sci.*, 7, p. 525; 1917, and B. S. Bulletin 14, p. 591; 1918.

It is found that numerous solid substances which have a low electrical conductivity in the dark have a high electrical conductivity when exposed to light, selenium being the most widely-known example. This photosensitive property is a function of the temperature of the substance; also of the intensity and of the wave length of the thermal radiation stimulus.

II. SELECTIVITY OF REACTION TO WAVE LENGTH OF RADIATION STIMULUS.

Using an equal energy stimulus, it is found that the photo-electrical response of substances, in general, is fairly uniform throughout the visible spectrum, terminating in a band or bands of high photo-electrical sensitivity in the extreme red [2] or near infra-red spectrum at 1.2μ ($\mu = 0.001$ mm.). Molybdenite is unique in having three bands of spectro-photo-electrical sensitivity which extends to 2μ. No substance is known which reacts photo-electrically to radiation of wave length greater than 3μ.

It is of interest to compare the distribution of energy in the visible spectrum of the Nernst glower as registered by means of a selenium cell and by a thermopile. The former was used in connection with a d'Arsonval galvanometer and the latter with a Thomson galvanometer. Fig. 1, Curve *A*, gives the spectral energy distribution obtained with a bismuth-silver thermopile (a non-selective radiometer), while Curve *B* gives the response of a selenium cell when similarly exposed (for 10 seconds) in different parts of the spectrum. After exposure to the low intensities, in the blue-violet, it required 20 to 30 seconds for the galvanometer reading to return to zero; and after exposure to the highest intensities it required over 2 minutes for the cell to return to its initial resistance. The curve obtained with the selenium cell is entirely erroneous as regards the actual spectral energy distribution.

Curve *C* gives [3] the response that is obtained by exposing parts of the spectrum using a high intensity, while Curve *D* represents the response at one-sixteenth the intensity used to obtain Curve *C*.

III. SELECTIVITY OF REACTION TIME.

It is well known that the photo-electric response of selenium lags considerably more for infra-red rays than for visual rays.

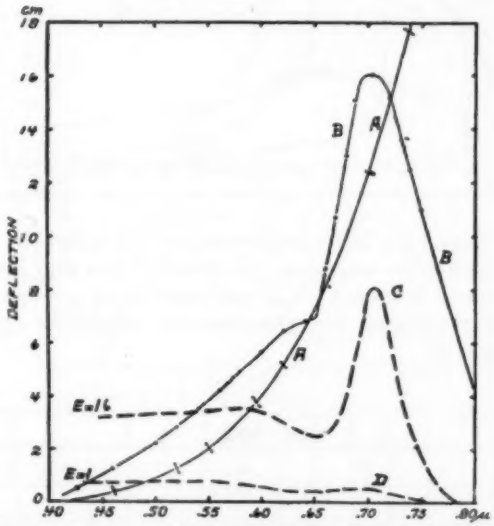
[2] Cuprous oxide appears to have its maximum sensitivity in the ultra-violet; Pfund, *Phys. Rev.* 7, p. 289; 1916.

[3] Pfund, *Phys. Rev.*, 34, p. 370; 1912.

Molybdenite and Case's thalofide cell are much quicker than selenium in their photo-electric response (change in electrical conductivity) when exposed to radiation, irrespective of wave-length.

In the visible spectrum the photo-electrical reaction attains equilibrium within a few seconds after exposure. On the other hand, it requires from one to several minutes' exposure to infra-red rays in order to obtain the maximum photo-electrical action.

FIG. 1.



Distribution of energy in the spectrum of a Nernst glower as observed with a non-selective radiometer, *A*, and with a selenium cell, *B*. Curves *C* and *D* (from Pfund) give the responses of a selenium cell in an equal energy spectrum.

This is shown in Fig. 2, which illustrates the photo-electrical behavior of molybdenite. Decreasing the temperature appears to shorten the time required for attaining the maximum photo-electrical response for all wave lengths.[4]

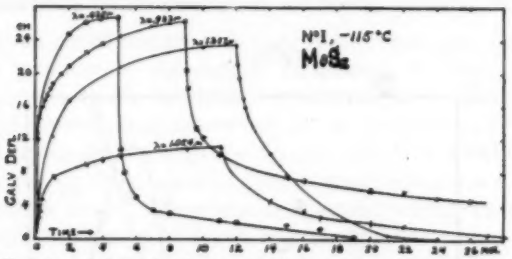
In making photo-electrical observations it is important to expose the substance for a sufficient time to obtain the maximum reaction. This is illustrated in Curve *A*, Fig. 3, which gives the

[4] *Bul. Bur. Stds.* 15, p. 132; 1919.

spectro-photo-electrical sensitivity of molybdenite when exposed for 10 seconds, while Curve *B* gives the corresponding curve for unlimited exposure; that is, exposure until there is no further increase in the galvanometer deflection.

At $\lambda = 0.76\mu$ the difference in photo-electric current (galvano-

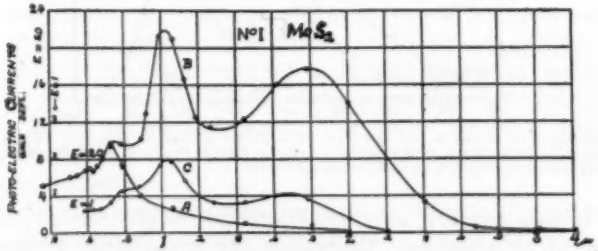
FIG. 2.



Variation of photo-electric current with time of exposure to radiation.

meter deflection) for limited exposure and for unlimited exposure (which at this point was about 30 seconds) was very small. On the other hand, beyond $\lambda = 1.4\mu$ the response is so slow that, on a 10-second exposure, the photo-electric sensitivity appears to be inappreciable.

FIG. 3.



Effect of intensity of radiation upon spectral photo-electric sensitivity.

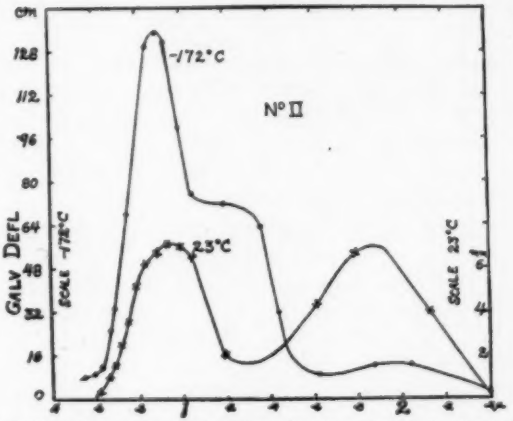
IV. SELECTIVITY OF REACTION TO INTENSITY OF RADIATION STIMULUS.

Using an equal energy spectrum, and different intensities, it is found that the resulting change in electrical conductivity or reaction is selective with respect to the wave length of the radiation stimulus.

As illustrated in Curves *C* and *D*, Fig. 1, and Curves *B* and *C* in Fig. 3, the effect of increasing the radiation stimulus (intensity E = 1 = 16 = 20) appears to produce a greater response in the long wave lengths than in the short wave lengths, with a resultant shift of the maximum of the spectral photo-electric sensitivity curve towards the long wave lengths.[5]

There does not appear to be a simple law governing the varia-

FIG. 4.



Effect of temperature upon spectral photo-electric sensitivity of molybdenite.

tion in the photo-electric response with variation in intensity of the radiation stimulus.

V. SELECTIVITY OF REACTION TO TEMPERATURE OF SUBSTANCE.

The temperature of the substance has a very marked effect upon the spectro-photo-electrical reaction of a substance. Reducing the temperature to a low degree, by immersion in liquid air, the substance is practically an electrical insulator. Nevertheless, on exposure to radiation of certain wave lengths, the electrical conductivity is greatly increased.

A typical example is that of molybdenite, illustrated in Fig. 4.

These curves show that, as the temperature decreases, the radiations of short wave lengths produce a greater change in

[5] See further data on silver sulphide, *Bul. Bur. Stds.* 15, p. 208; 1919.

electrical conductivity than do the long wave lengths. As a result, the maximum of the sensitivity curve is shifted towards the short wave lengths. Furthermore, the maxima are generally very much sharper at low temperatures as found by Elliot [6] for selenium, and by the writer [7] for silver sulphide.

VI. SUMMARY.

In the foregoing pages attention is called to the fact that numerous solid substances, which have a low electrical conductivity in the dark, have a high electrical conductivity when exposed to light.

A study of some of these substances shows that their photo-electrical reactions have the following characteristics:

1. The photo-electrical response, to an equal energy stimulus, is fairly uniform throughout the visible spectrum, terminating in a band or bands of high photo-electrical sensitivity in the extreme red or near infra-red spectrum.

2. The time required to attain equilibrium of the photo-electrical reaction is selective to wave length. For visible radiation the photo-electrical response is complete in a few seconds; whereas, it requires an exposure of one to several minutes to infra-red rays in order to induce the maximum photo-electrical response.

3. The effect of increasing the intensity of the radiation stimulus is to produce a greater increase in the photo-electric response to the long wave lengths than to the short wave lengths, with a resultant shift of the maximum of the spectral photo-electric sensitivity curve towards the long wave lengths.

4. Decreasing the temperature of the substance greatly increases the intrinsic photo-electrical sensitivity throughout the whole spectrum and shifts the maximum sensitivity towards the short wave lengths.

WASHINGTON, D. C.
June 25, 1920

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[6] Elliot, *Phys. Rev.* (2) 5, p. 59; 1915.
[7] *Bul. Bur. Stds.* 15, p. 242; 1919.

ON THE RELATION BETWEEN PHOTOGRAPHIC DENSITY, LIGHT INTENSITY, AND EXPOSURE TIME.*

BY

FRANK E. ROSS.

IT is a matter of experience that the degree of blackening of the photographic plate depends upon the intensity I, the wavelength λ, and the time of action t, of radiation falling upon it. This effect, which, from its analogy with a somewhat similar phenomenon taking place in vision, may be called "photographic sensation," is measurable physically as transparency T, or opacity O, where $O = T^{-1}$. It has been found convenient in photography to use a derived quantity called "density," defined by

$$D = \log_{10} O. \tag{1}$$

There are several reasons making density a desirable measure of photographic sensation. It is found, for instance, that a series of equidistant densities, or densities increasing in arithmetical progression, produce visually a sensation or appearance of *equal steps;* in other words, visual response is, within certain limits, a linear function of density (Fechners' Law). Again, it is found experimentally, that within wide limits the relation between mass (M) of reduced silver per unit area (cm.^{2}) and density is linear, of the form

$$D = c\,M. \tag{2}$$

c is called the photometric constant. Its value is approximately 0.10 milligrams for ordinary photographic negatives. It is assumed that the viewing illumination in measuring density is completely diffuse.

It can be assumed as a matter of experience that

$$D = F\,(I, t, \ldots), \tag{3}$$

or, density is a function of intensity, time of action, and other variables or parameters. These parameters are so numerous that it is impracticable even to specify them. For the present the reciprocity law will be assumed to hold, namely, that I and t occur in (3) only in a product form.

*Communication No. 93 from the Research Laboratory of the Eastman Kodak Company.

Neglecting all formulæ purporting to take account of the phenomenon of "reversal," four specific formulæ of the form (3) have been proposed: (a) by Abney [1] in 1889; (b) by Hurter and Driffield [2] in 1890; (c) by Elder [3] in 1893; (d) by Channon [4] in 1906. Abney's formula is purely empirical. The formulæ of Hurter and Driffield, of Elder, and of Channon are based upon differential equations specifying the velocity of the light action, or the velocity of "infection" of the silver grains, and are to that extent to be considered as theoretical equations.

The formula of Abney is

$$T = e^{-\mu\left(\log\frac{It}{i}\right)^2}, \tag{4}$$

or, in terms of density,

$$D = \mu' \log^2 \frac{It}{i}. \tag{4a}$$

An example is given in Figure 1 of the character of the agreement to be expected between Abney's formula (4) and observed curves. In the diagram, transparency is plotted against the logarithm of the exposure It. The full curves are of a commercial rapid plate, developed for one and three minutes respectively. The dotted curves are computed from Abney's formula, the constants μ and i being so chosen as to produce the closest possible agreement with the measured curves. The values of the constants are:

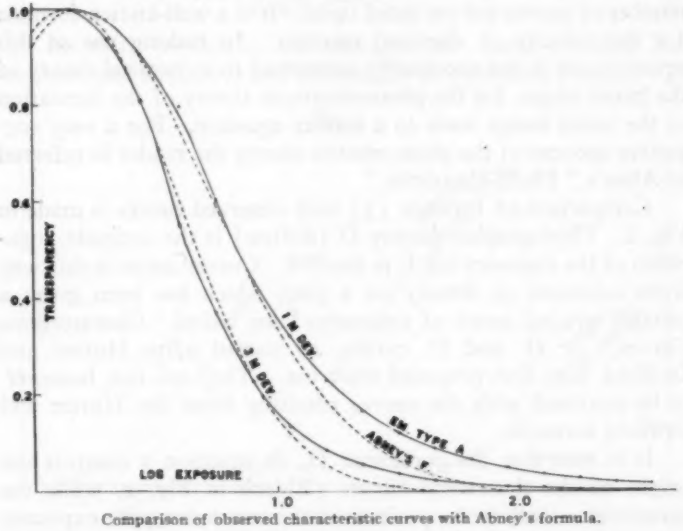
	μ	$\log i$	γ
1m. development	1.59	0.179	0.91
3m. development	2.81	0.176	1.24

The quantity γ in the last column refers to the measured curves, and will be explained presently. It will be noticed that i is independent of, while μ varies with, development time. i can be taken as a measure of the speed of a plate, while μ is a measure of the contrast value. Agreement of observed and computed curves is fairly good from transparency unity to $T = 0.30$ ($D = 0.52$) in the case of one minute development, and as far as $T = 0.15$ ($D = 0.82$) for three minutes development. The formula has no value for lower transparencies, corresponding to higher densities. In fact, Abney's formula can be considered to give a very good

[1] *Journ. Camera Club*, April, 1889, and April, 1893.
[2] *Journ. Soc. Chem. Ind.*, May, 1890.
[3] *Journ. Camera Club*, April, 1893.
[4] *Photographic Journ.*, June, 1906.

representation of the relation between transparency or density, and exposure, for the first step in the process, which covers the appearance of the image and its development in strength up to the point where rapid action begins. For certain character of work making use of weak or so-called under-exposures, Abney's formula applies more accurately than any other which has been proposed. There does not appear, however, to be any theoretical

FIG. 1.



Comparison of observed characteristic curves with Abney's formula.

basis for it, at least none has been proposed. It fails to take account of the very obvious fact that there is a limit to the density which can be reached depending upon the amount of silver in the emulsion, the relation being given by equation 2 above.

On account of its simplicity, Elder's formula will be considered next, although preceded in point of time by the more complicated Hurter and Driffield formula. Elder's equation is

$$D = D_m\left(1 - e^{-\kappa It}\right), \tag{5}$$

which is derived by integration of the differential equation

$$dn = \kappa I(N - n)dt, \tag{5a}$$

where D_m is the maximum density obtainable, and is a function of the amount of silver in the emulsion; N is the total number of grains per cm.^{2}, to which D_m is proportional; n is the number of grains changed to latent image in the time $t = 0$ to $t = t$; I is light intensity; κ is a parameter depending principally upon grain-sensitivity and wave-length of light, and is a measure of the speed of the emulsion.

Equation (5a) asserts that the number of grains changed to latent image in any small interval of time is proportional to the number of grains not yet acted upon. It is a well-known formula for the velocity of chemical reaction. In making use of this equation one is not necessarily committed to a chemical theory of the latent image, for the photo-electronic theory of the formation of the latent image leads to a similar equation. For a very suggestive account of the photo-electric theory the reader is referred to Allen's "Photo-electricity."

Comparison of formula (5) with observed curves is made in Fig. 2. Photographic density D (diffuse) is the ordinate, logarithm of the exposure log E is abscissa. Curves made in this way from measures of density on a plate which has been given a suitably graded series of exposures[5] are called "Characteristic Curves," or H. and D. curves, so named after Hurter and Driffield, who first proposed their use. They are not, however, to be confused with the curves resulting from the Hurter and Driffield formula.

It is seen that the parameter D_m in equation 5 controls the height of the theoretical curves (Elder) in Fig. 2, while the parameter κ controls the position of the curve along the exposure axis, and is accordingly a measure of the speed of the emulsion. D_m and were chosen in each case to agree with the maximum density and average position along the exposure axis of the observed curves, thus giving the closest possible agreement. It will be seen that the agreement is not at all good, but is better in the case of the emulsion of Type B than in that of Type A.

These emulsion types need a few words of explanation. A characteristic curve consists of three parts: (1) A portion of low density and slow change in density, called the "toe" of the curve;

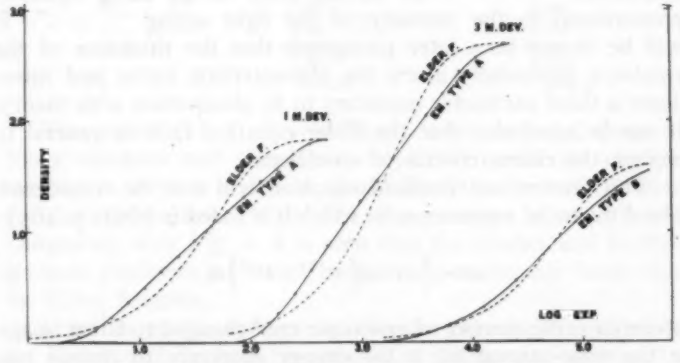
[5] Instruments in which such exposures are made are called "Sensitometers." For a description, see paper by L. A. Jones, "A New Non-Intermittent Sensitometer." *J. Frank. Inst.*, 1920, p. 303.

(2) a straight, or nearly straight portion comprised between the regions of low and high density, in which the *change* of density reaches a maximum, called the straight-line portion. The maximum slope is called gamma (γ) and is an important constant in an emulsion, as it determines the contrast or power to reproduce shades in lighting. It is the factor controlling sensitivity in photographic photometry. Mathematically, gamma is given by

$$\gamma = \left(\frac{dD}{d \log E}\right) \text{Max.}; \qquad (6)$$

(3) the third portion of the characteristic curve is called the shoulder, and lies above and beyond the straight-line portion. It

FIG. 2.



Comparison of observed characteristic curves with Elder's formula.

is a region of high density and slow change of density. Briefly, the toe is the region of under-exposure, the straight-line portion is the region of correct exposure, and the shoulder is the region of over-exposure.

An emulsion of Type A is one having a very short "toe." Type B is one having an extended toe. Similar differences are to be found in the shoulder, but generally they are not of such importance. These differences in the toe in emulsions of various types are of importance in the subjects of photographic resolving power and sharpness.

If equation 5a is a complete description of the formation of the latent image, assuming regular and proportional development,

computed and observed curves in Fig. 2 ought to coincide. As already remarked, this is found not to be the case. Only two parameters are at our disposal in equation 5. A third parameter appears to be necessary. At the time the equation was proposed, the question was raised whether it should not be modified to take account of the thickness of the emulsion. Elder [6] advanced what purported to be a proof that the equation as it stands virtually takes account of film thickness, it being only necessary to multiply the light intensity in equation 5a by the factor $\frac{a^s-1}{a \log e}$, where a is the absorption coefficient of the film and s the thickness. Accordingly the form of equation 5 should be unaltered if his demonstration is correct. The fallacy in the proof lies in the statement, " . . . the action, other things being equal, is proportional to the intensity of the light acting . . ." It will be shown in a later paragraph that the thickness of the emulsion profoundly alters the characteristic curve and introduces a third parameter necessary to fit observation with theory. It can be concluded that the Elder equation fails in general to explain the characteristics of emulsions.

The Hurter and Driffield equation will now be considered. The differential equation upon which it is based is (*ibid.*, p. 465):

$$dx = \frac{I}{b}(1-r)\left[e^{-\kappa x} - e^{-\kappa a}\right] dt, \tag{7}$$

where dx is the number of grains per cm.^{2} changed to latent image in the time-interval dt; b the energy necessary to change one grain; I the light intensity; r the reflection coefficient; a the total number of grains per cm.^{2} in the emulsion; κ is a parameter to be considered later. The integral equation of (7) is easily found. Combining parameters and constants and making use of equation 2, there results

$$D = \rho \log_{10}\left[O-(O-1)e^{-q\,It}\right], \tag{8}$$

where $O = e^{-\kappa a}$. There are three parameters in (8), one more than in equation 5, so that it can be expected to give closer agreement with observed curves than is possible in the case of the Elder equation.

[6] *J. Camera Club*, July, 1893, p. 166.

Equation (8) will now be applied to the two emulsion types A and B used in previous comparisons. In fitting equation 8 to plotted curves, the parameters O and ρ are determined by the following formulæ:

$$\left.\begin{aligned} \rho \log_{10} O &= D_m, \\ \rho &= \frac{\gamma}{1-e^{-\log O}}. \end{aligned}\right\} \qquad (9)$$

The parameter q is determined by making the mid-point of the straight-line portion a common point of the observed and computed curves. The constants and parameters found in this way are as follows:

	Em. A	Em. B
D_m	2.70	1.60
γ	1.34	1.00
O	40.46	10.72
ρ	1.675	1.555
q	.00127	.00235

The curves (H. D. F.) in Fig. 3 were computed from the above numbers and plotted in conjunction with the observed curves. In the case of Em. A, disagreement in the toe and shoulder is very marked. With Em. B agreement is much better. Comparing with Fig. 2, it is seen that the Hurter and Driffield formula conforms to actual emulsions considerably better than the Elder formula.

It will be worth while to examine the assumptions underlying the Hurter and Driffield formula. The differential equation (7) is based on energy relations. The reasoning is as follows: If I is the intensity of the light incident on the plate, corrected for reflection loss, and T the transparency of the emulsion, the amount of energy absorbed by the plate in unit time is obviously

$$E_a = I(1-T_a), \qquad (10)$$

it being assumed that there are a grains of silver halide per cm.^{2} in the emulsion. The subscripts in (10) and following formulæ are simply descriptive of the particular mass of grains considered.

Let the number of grains changed to latent image after a time t be denoted by x. Call the transparency of this group of grains T_x. The energy absorbed by this group in unit time is said to be

$$E_x = I(1-T_x). \qquad (11)$$

Let the amount of energy necessary to change one grain of silver bromide to latent image be denoted by b. It is claimed that the number of grains dx so changed during the time dt will be given by the relation

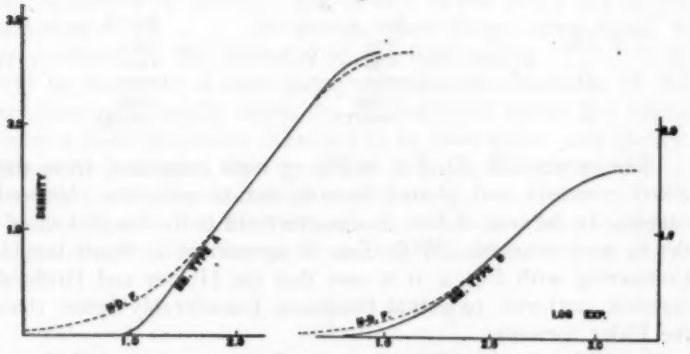
$b.dx$ = amount of energy absorbed by the *unchanged* silver grains; (12)

from which

$$b.dx = (E_a - E_x)dt. \tag{12a}$$

The truth of equation (12) can not be admitted. It is not clear how the energy absorbed by the group of grains $a-x$ can be applied to the sub-group dx.

FIG. 3.



Comparison of observed characteristic curves with Hurter and Driffield's formula.

Continuing, the next step is the substitution of the following values in (12a):

$$T_a = e^{-kt}, \tag{13}$$

$$T_x = e^{-kx}. \tag{13a}$$

In writing equation (13a) serious error is made. Equation 13 is undoubtedly valid, at least as an approximate equation. If (13a) is true, it is equally true with regard to the group of grains $a-x$, *i.e.*,

$$T_{a-x} = e^{-k(a-x)}; \tag{14}$$

and equally with (11)

$$E_{a-x} = I(1 - T_{a-x}). \tag{15}$$

We have, however,

$$E_a = E_x + E_{a-x}, \tag{16}$$

or,

$$I(1-T_a) = I(1-T_x) + I(1-T_{a-x});$$

substituting the assumed values of T there results

$$e^{-x_a} - e^{-x_x} = e^{-x_a} e^{-x_x} - 1,$$

which is true only for $x = o$. The assumption of Hurter and Driffield that the transparency of the x-group of grains is given by e^{-x_x} is accordingly incorrect, for it leads to a mathematical inconsistency. Their equations (7) and (8) in consequence must be considered purely empirical.

Channon's Formula: Channon follows Elder in using the mass-action equation, but takes into account emulsion thickness. The equation to which he is led is not directly integrable. By expansion in series his solution takes the form

$$D = \gamma \left\{ \phi(aE) - \phi(aE\vartheta) \right\} \tag{17}$$

where γ, a, and ϑ are parameters, E is exposure, and ϕ a functional form as follows:

$$\phi(n) = n - \frac{n^2}{2\lfloor 2} + \frac{n^3}{3\lfloor 3} - \frac{n^4}{4\lfloor 4} + \ldots$$

Agreement with observed curves is of the same order as the H. and D. formula. Theoretically, it is well grounded, but fails to take account of varying sensitivity of grain.

Development of a New Formula.—On account of the general applicability of the mass-action equation to certain classes of chemical and physical problems, it seems worth while to apply it to the present problem in a manner more circumspect and cautious than has hitherto been done. It will be assumed that the law applies only to the case of a single layer of grains, which therefore individually receive equal exposures. Actual emulsions are composed of grains lying in layers many layers thick and very often of grains of varying sensitiveness. Obviously many formulæ can be constructed, depending upon particular assumptions of the following classes:

a. Relative frequency of grains of varying size.

b. Relative sensitivity of grains of varying size.

c. Variation of grain-sensitivity with light-intensity and wave-length for each class of grains.

d. Ratio of inert to active grains, depending on grain size.

e. Number of layers of silver halide.

f. Opacity of the emulsion and its variation with wavelength.

This is not an exhaustive enumeration of the factors which must be considered in developing the theory of the subject. It would be unprofitable in the absence of data on the controlling factors enumerated above to develop a formula on any but the simplest assumptions. These are:

1. All the grains in an emulsion divisible into n groups, each group as a whole obeying the mass-action law.

2. The silver mass the same for each group.

3. The sensitivity factor ( ) of the groups arrangable in geometrical progression. This takes account of the factors of: (a) true sensitivity differences, (b) apparent sensitivity differences, due to the thickness of the emulsion. This equivalence is possible on account of the reciprocal equivalence of κ and I in formula (5a).

With these assumptions, the equation connecting density with exposure can easily be derived. Calling d_m the maximum density (for infinite exposure); κ the sensitivity factor; r the common ratio of the sensitivities of the various groups, the equation becomes

$$D = d_m \left[1 - \frac{1}{n} \sum_{s=0}^{s=n-1} e^{-\kappa r^s It} \right]. \qquad (18)$$

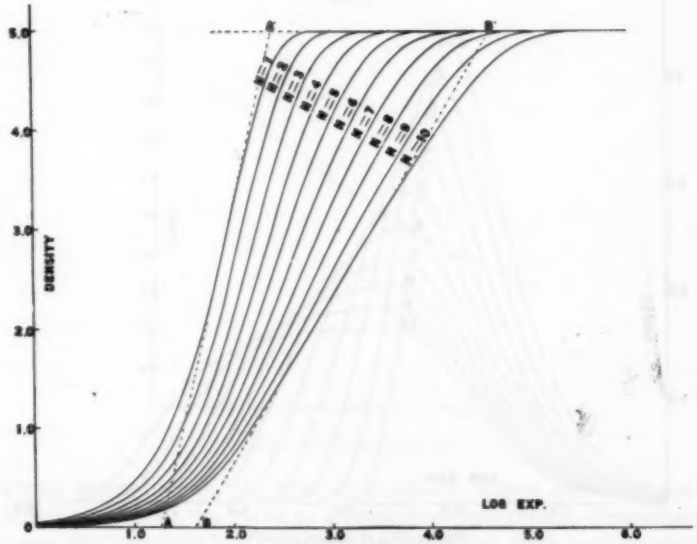
In order to test formula (18), it will be necessary to give values to the parameters r and n. If the values $r = \frac{1}{2}$ and $n = 1, 2, 3 \ldots 10$ are assigned, the field of practical emulsions should be well covered. For, neglecting real sensitivity differences, the opacities, including real sensitivity differences, of our ideal emulsions would in this case vary from unity to $2^9 = 512$, a variation more than sufficient to include all emulsions.

Fig. 4 shows a series of curves calculated from equation 18, for $r = \frac{1}{2}$, and $d_m = 5$, for successive values of $n = 1$ to $n = 10$. The curves for any other value of d_m can be obtained from these curves by multiplying the ordinates by $\frac{1}{5}\, d_m$. It will be noticed that for increasing values of n the contrast or gamma becomes less and the straight-line portion longer. The "toe" diminishes with

n, a characteristic not to be found in the case of the Hurter and Driffield equation.

In photographic literature the "latitude" of an emulsion is defined as the antilogarithm of the projection of the straight-line portion of the characteristic curve on the log-exposure axis. It will be convenient to define a term "range," of similar properties to latitude, in order to meet the objection that a mathematical curve has no true latitude. By "range" is meant the antilogarithm of the projection of the line AA′ or BB′ (Fig. 4) on

FIG. 4.



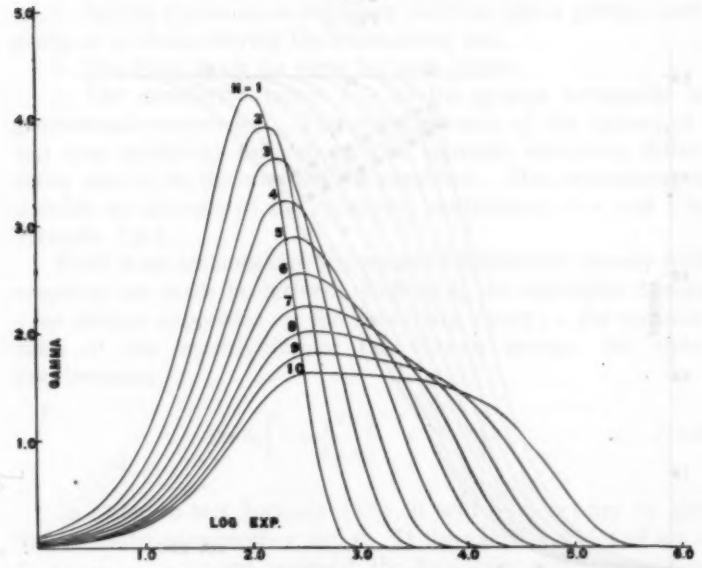
Characteristic curves from new formula.

the exposure-axis. It can be taken as a measure of the reproducing power of an emulsion and is accordingly a useful and important function. By measurements made on Fig. 4 it is found that the range has the following values:

n	*Range*	*n*	*Range*
1	15	6	90
2	17	7	160
3	25	8	290
4	35	9	530
5	55	10	1000

Fig. 5 shows the first-derivative curves for the same series of emulsions, obtained by differentiating equation (18) with respect to the logarithm of the exposure. The maximum ordinate of each curve is the gamma of the corresponding emulsion. From the broadening out of the curves with increasing n it is seen how the range increases, and gamma decreases, with emulsion thickness. The relation between gamma and n can be empirically derived from this series of curves. However, gamma derived

FIG. 5.



Slope curves from new formula.

from Fig. 5 will be found slightly greater than if obtained from the straight-line portion of the curves in Fig. 4. In comparing theory with observed curves, it will be better to use values obtained from Fig. 4. They have accordingly been scaled off as accurately as possible, plotted, and a smooth curve drawn. The result, an exponential curve, is shown in Fig. 6 reduced to $d_m = 1$.

It remains to apply formula (18) to the emulsion types A and B, which have been used in the previous comparisons. This can be easily done with the help of Fig. 6. From the characteris-

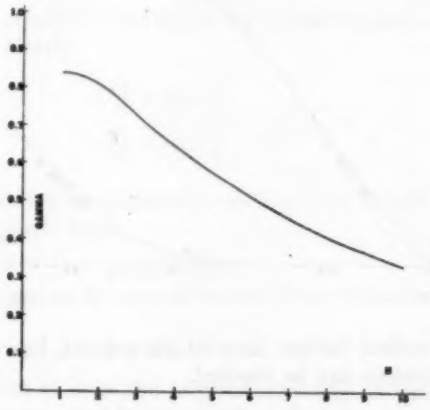
tic curves of A and B, γ and d_m, are known. With the ratio as ordinate, n is obtained directly from the figure. We have:

	γ	d_m	γ/d_m	n
Em. A.....	1.34	2.70	0.496	6
Em. B.....	1.00	1.60	0.625	4

Scale off the ordinates of the curves in Fig. 4 for $n = 6$ and for $n = 4$. Multiply each value by $d_m/5$. The result is the theoretical characteristic curves, which are given in Fig. 7. Agreement with the observed curves is seen to be very good in the case of Type B.

It is quite evident that there is an appreciable secondary

FIG. 6.

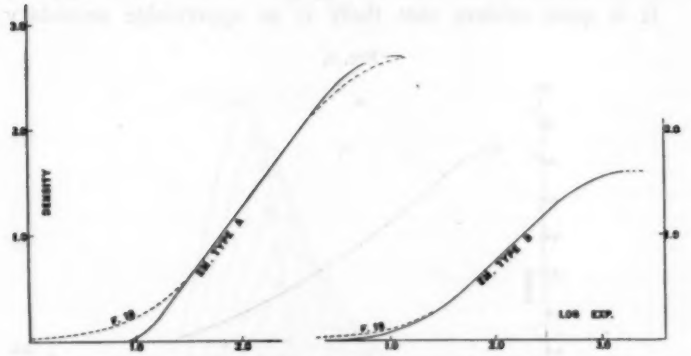


Variation of Gamma with n.

action depending on the type of emulsion, which depresses the initial portion of the curve to a greater or less extent, depending upon the emulsion. Translated in terms of light action, this means that there is an inertia period—possibly a chemical change of some sort produced by the light is necessary before the normal action is effective. The other alternative is that it is a developer effect. There is evidence that the latent image forms at numerous points of the silver halide grain, which increase in number and size with exposure. To explain the observed depression of the curve it is only necessary to assume that a grain does not fully develop until the latent image connected with it has reached a certain phase. The effect must vary with the emulsion. It is

necessary to state here that the observed characteristic curves shown in this paper have all been obtained from a time-scale, or t variable. Curves resulting from an intensity-scale are not at hand. They should coincide with the time-scale curves if the reciprocity law holds. If Schwarzschild's law holds, they will differ in the density-ordinate only by a common factor p, and must be accordingly of the same form, as will be shown. It is

FIG. 7.



Comparison of observed characteristic curves with new formula.

necessary to collect further data on the subject, however, before definite conclusions can be reached.

THE RECIPROCITY LAW.

The early experimenters found that the same photographic effect could be obtained by reciprocal variations of light-intensity and exposure-time. The relation, therefore, assumed the comparatively simple form

$$D = F(E);\ E = It.$$

It will be useful to assume that density D can be expressed as a function F of a single variable U and then find from experimental data the relation between U and the variables I and t. With this assumption

$$D = F(U), \qquad (21)$$
$$U = \phi(I; t).$$

Differentiating,

$$\delta D = \frac{dF}{dU}\left[\frac{\partial \phi}{\partial I}\delta I + \frac{\partial \phi}{\partial t}\delta t\right];$$

or, expressed in logarithmic increments,

$$\delta D = \frac{dF}{dU}\left[I\frac{\partial \phi}{\partial I}\delta(\log I) + t\frac{\partial \phi}{\partial t}\delta\ (\log t)\right]. \tag{22}$$

Assume that a pair of equal densities D_0 are produced by two pairs of values of the variables I and t; *i.e.*,

$$D_0 = F(U_0),$$
$$U_0 = \phi(I_1; t_1) = \phi(I_2; t_2).$$

What is the condition that δD in (22) shall depend only on U or on ϕ? It is clearly

$$\left.\begin{aligned} t\frac{\partial \phi}{\partial t} &= \psi(\phi), \\ I\frac{\partial \phi}{\partial I} &= \chi(\phi). \end{aligned}\right\} \tag{23}$$

If a power series be assumed to satisfy (23) it will be found to reduce to a single term

$$\phi = I_m^m . t^n = I . t^p, \tag{24}$$

or alternatively

$$\phi = \log\ (I^u t^v). \tag{24a}$$

It is important to state the physical meaning of equations (23). If these equations hold, the variables I and t do not occur explicitly in the coefficients in the right-hand member of equation 22. In the case assumed, of two equal densities D_0 produced by the same value of U but different values of I and t, the increments δD which they undergo from equal increments of log I or of log *t* accordingly must be identical. In order that this proposition may extend to finite values of density change, it is further necessary that $t^n \frac{\partial^n \phi}{\partial t^n}$ and $I^n \frac{\partial^n \phi}{\partial I^n}$ shall be functions of ϕ. Formulæ (24) and (24a) are seen to satisfy this criterion. The interpretation is as follows:

Suppose a series of densities $D_0, D_1 \ldots D_m$ are produced by a series of exposures in which either the time or the

intensity increases in geometrical progression (logarithmically), *i.e.*,

$$\begin{array}{ll} \text{time or intensity} \ldots\ldots & \left\{\begin{array}{lll} t & 2t & 4t \\ I & 2I & 4I \end{array}\right\} \\ \text{density produced} \ldots\ldots & D_0 \quad D_1 \quad D_2 \end{array}$$

Suppose another series produced in the same way, with this difference, that the initial density D_0' is produced by an entirely different combination of I and t than obtains in the first series. We have then

$$\begin{array}{ll} \text{time or intensity} \ldots\ldots & \left\{\begin{array}{lll} t' & 2t' & 4t' \\ I' & 2I' & 4I' \end{array}\right\} \\ \text{density produced} \ldots\ldots & D_0' \quad D_1' \quad D_2' \end{array}$$

The theorum just proved asserts that if $D_0 = D_0'$, we must have

$$\left.\begin{array}{l} D_1 = D_1' \\ D_2 = D_2' \\ \ldots\ldots \\ \ldots\ldots \end{array}\right\} \tag{25}$$

Or, in words, a series of densities produced by logarithmic variations of I or t is independent of how the initial density in the series is produced. Expressed otherwise, if two series of densities are produced in this way, if any one member of one series coincides in density with any one member of the other series, then the two series will coincide throughout.

To what extent are equations (23) and (24) realized in practice? The earlier experimenters found the very simple expression

$$\phi = I.t, \tag{26}$$

which satisfies (23). Schwarzschild and others have since found

$$\phi = I.t^p, \tag{27}$$

which also satisfies (23). On the other hand, E. Kron [7] has found

$$\phi = t.I.10^{-a\sqrt{\left(\log \frac{I}{I_0}\right)^2 + 1}}, \tag{28}$$

in which a and I_0 are parameters depending upon the emulsion. Applying (23) to this equation gives

$$t\frac{\partial \phi}{\partial t} = \phi,$$

$$I\frac{\partial \phi}{\partial I} = \phi\left\{1 - \frac{a}{\text{Mod}} \log \frac{I}{I_0}\left[\left(\log \frac{I}{I_0}\right)^2 + 1\right]^{-\frac{1}{2}}\right\}. \tag{28a}$$

[7] "Ueber das Schwarzungsgesetz photographischen Platten" Eder's Jahrbuch, 1914, p. 6.

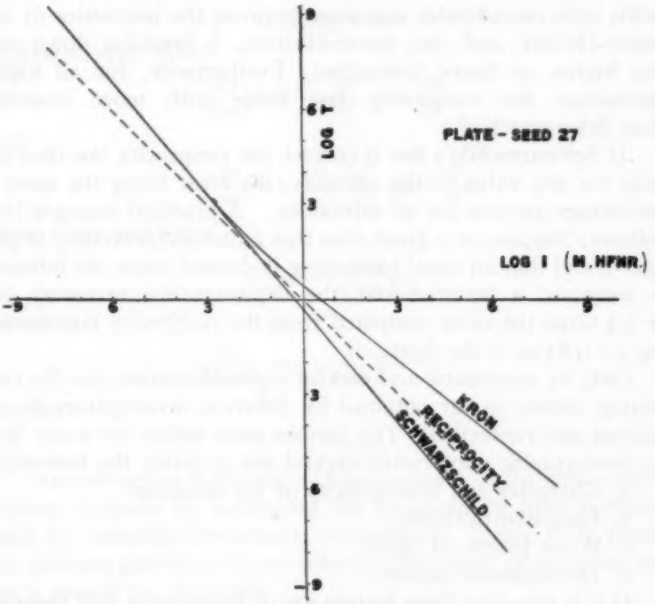
Accordingly only the first of (23) is satisfied. This means that according to Kron's experimental results the time-scale holds, but not the intensity-scale. In other words, the two series of densities considered above

$$D_0 \quad D_1 \quad D_2,$$

and

$$D_0' \quad D_1' \quad D_2',$$

FIG. 8.



Curves of equal blackening.

are equal for time-variations, but not for intensity-variations. This is a departure from results so far obtained and should be confirmed by numerous experiments.

The relation between log I and log t to produce equal blackening in the respective cases of the reciprocity law, of Schwarzschild's, and of Kron's law holding is plotted in Fig. 8 for Seed

27 plates, for which the parameters a and $\log I_0$ are given by Kron. They are:

$$a = 0.21; \; I_0 = .013 \text{ meter-Hefners.}$$

Kron shows, which can easily be verified, that his law becomes identical with Schwarzschild's for low intensities, the relation between the parameters being

$$p = \frac{1}{1+a}.$$

From the figure this is seen to be true. Study of the curves is instructive. If Kron's results are correct, the reciprocity law holds with considerable exactness between the intensities of one meter-Hefner, and .001 meter-Hefners, it breaking down only for higher or lower intensities. Furthermore, for all higher intensities, the reciprocity law holds with more exactness than Schwarzschild's.

If Schwarzschild's law is correct, the reciprocity law does not hold for any value of the intensity, the error being the same in percentage amount for all intensities. A practical example is as follows: Suppose in a given case that a certain blackening is produced, and that an equal blackening is desired when the intensity is increased a hundred-fold, the exposure-time necessary will be 2.4 times the value computed from the reciprocity law, assuming $p = 0.83$ as in the figure.

Only by systematic and careful experimentation can the conflicting results so far obtained by different investigators be explained and reconciled. The factors upon which the exact laws of photographic photometry depend are probably the following:

a. Character and homogeneity of the emulsion.
b. Emulsion thickness.
c. Wave length of light.
d. Development factors.

If it is true that these factors are of importance, and theoretical consideration indicates that they are, it is not surprising that concordant results have so far not been obtained.

So long as the reciprocity law of photographic action was supposed to hold, the character of the fundamental process of latent image formation appeared to be simple. Since energy absorbed is the product of intensity and time of action ($E = It$), the formation of the latent image appeared to be only a function of the energy absorbed by the silver grains. In other words,

exposure should be merely a matter of energy absorption. It was from this standpoint that Hurter and Driffield developed their equation (8) of photographic action.

There are two elegant methods available for determining the exponent p in equation 24 as follows: (1) measurements of density; (2) measurement of diameters of star images. A discussion of the second method will be contained in a following paper. It is based upon considerations too extensive to be treated here. The first method is based upon a simple formula, which can be derived as follows: Assume

$$D = F(U); \; U = It^p.$$

Differentiation gives

$$\frac{\partial D}{\partial \log I} = I \frac{dF}{dU} \frac{\partial U}{\partial I} = U \frac{dF}{dU}, \tag{29}$$

and

$$\frac{\partial D}{\partial \log t} = t \frac{dF}{dU} \frac{\partial U}{\partial t} = pU \frac{dF}{dU}; \tag{30}$$

from (29) and (30),

$$\frac{\partial D}{\partial \log t} = p \frac{\partial D}{\partial \log I}; \tag{31}$$

or since

$$\frac{\partial D}{\partial \log t} = \gamma_t; \; \frac{\partial D}{\partial \log I} = \gamma_I; \tag{32}$$

we have

$$\gamma_t = p\gamma_I.$$

Accordingly p is the ratio of the gammas of the characteristic curves obtained by increasing the exposure by time-increments and by intensity-increments, respectively. Since it is possible to measure gamma with considerable precision, an accurate value of p should be obtainable.

If Kron's law holds the characteristic curves obtained by intensity variations (intensity-scale) will vary, being functions of the intensity (see equation 28a). Study of characteristic curves obtained under varying conditions as to time and intensity therefore will give a direct indication of the law relating photographic effect to exposure-time and intensity.

EASTMAN KODAK COMPANY,
ROCHESTER, N. Y.,
May 14, 1920.

THE CEMENTED TELESCOPE OBJECTIVE OF BARIUM CROWN AND FLINT.

BY

I. C. GARDNER.

HARTING[1] showed that the most satisfactory pair of glasses available for a telescope objective of the two component cemented type is a barium crown, N_D 1.57 to 1.59 and an ordinary flint lying between 1.60 and 1.66. Glasses within this range offer the possibility of correcting simultaneously for chromatic and spherical aberration and the sine condition. When the war made necessary the manufacture of optical glass in this country the number of types to be manufactured was, for obvious reasons, made the smallest possible consistent with the demands to be satisfied. Five types were produced, and of these one was a barium crown, N_D 1.57, and a second was a flint, N_D 1.61 to 1.62. These two types were used in the numerous short focus, large aperture telescope objectives required for field glasses and other military optical instruments. Harting's paper had already standardized, to a considerable degree, the types of glass used in telescope objectives, and as outlined above, the emergency created by the war furthered the standardization. In view of this, and the fact that these glasses are now produced, having substantially similar characteristics, by the different American manufacturers, it is thought worth while to publish the following data and graphical representations which were obtained preliminary to more general and extended calculations and which show the possible combinations of values of the three aberrations which may be obtained by the use of barium crown and flint.

The following three glasses have been selected as typical and used throughout the computations:

	N_D	$N_D - N_C$	$N_F - N_D$	ν
Barium Crown No. 1	1.5740	.00299	.00718	56.4
Barium Crown No. 2	1.5740	.00286	.00713	57.5
Flint	1.6160	.00476	.01202	36.7

It is convenient to consider a two-parameter series of lenses in which one radius, say that of the cemented surface, is held con-

[1] Harting, Zeits. Instrumentenk. 18, p. 357, (1898).

stant and the two remaining radii varied. The refracting powers of the two outer surfaces, as defined by the following equations, will be selected as suitable parameters:

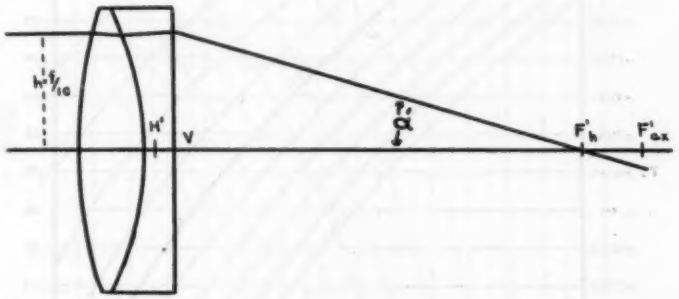
$$\varphi = \frac{n_1 - 1}{r_1} \qquad (1)$$

$$\psi = \frac{1 - n_2}{r_3}$$

The index of the crown component (placed facing the parallel light) is n_1, that of the flint n_2.

The linear aberrations may be measured in terms of the focal length. There is then no sacrifice of generality if we consider

FIG. 1.

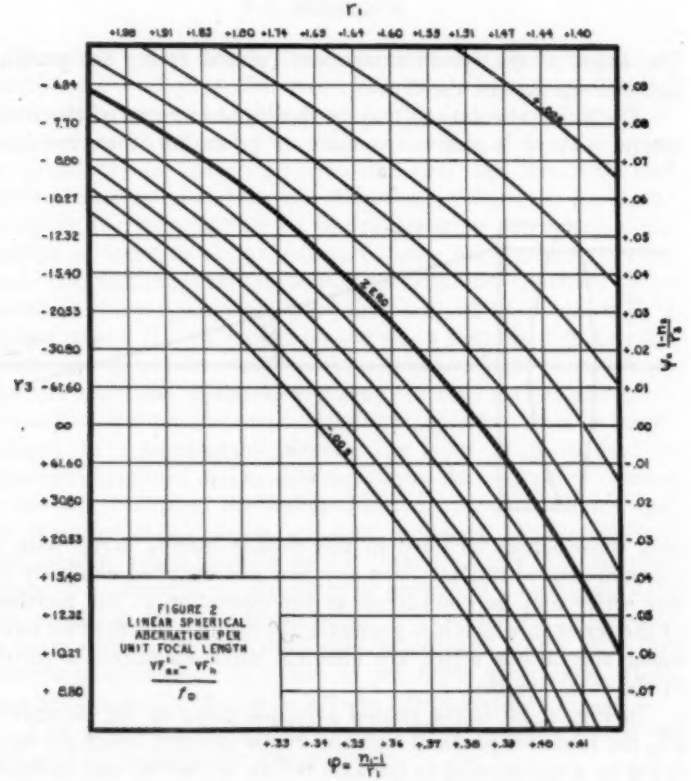


only those lenses included in this doubly infinite array with a constant second radius. For any lens not included explicitly in this series may be considered as corresponding to the member of the series to which it is geometrically similar and the two have linear aberrations which are identical when measured in terms of the focal lengths.

In Fig. 1, H′ is the second principle point of the objective, F'_{ax} the focus for paraxial rays and F'_h the point at which the axis is cut by a ray parallel to the axis before refraction and incident at a point distant one-tenth of the focal length from the centre of the lens. Then $VF'_{ax} - VF'_h$ is the linear spherical aberration. If f is the equivalent focal length for paraxial rays, $f - \frac{h}{\sin \alpha'}$ is the departure from the sine condition. The chromatic difference of equivalent focal lengths is $f_C - f_F$ where the subscripts denote that the equivalent focal lengths are to be taken for the C

and F spectrum lines. All of the above differences have been so selected that they are positive for an uncorrected converging lens.

We may measure φ and ψ along two coördinate axes as abscissæ and ordinates, respectively. Then any point φ_0, ψ_0 corre-



sponds to a lens having its radii defined by the equations

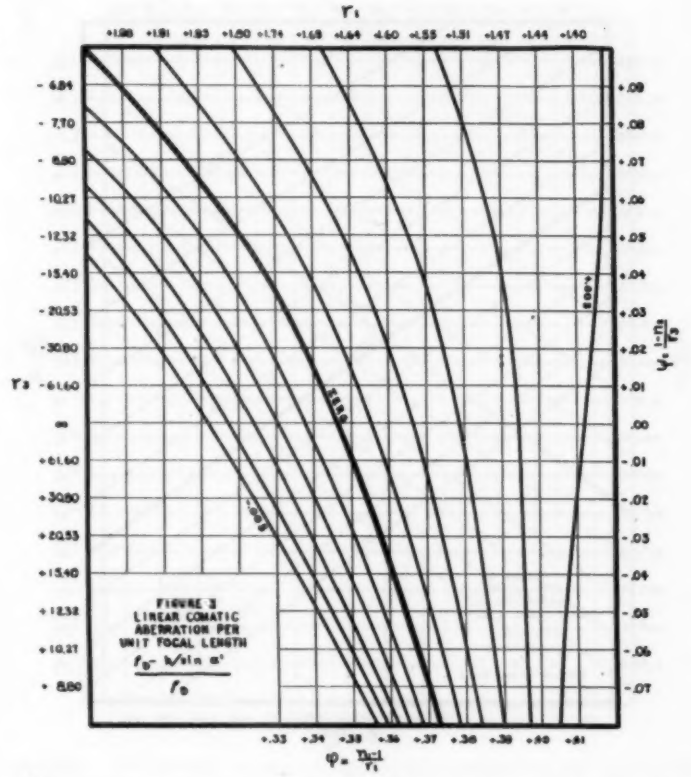
$$r_1 = \frac{n_1 - 1}{\varphi_0}$$
$$r_2 = -1 \qquad (2)$$
$$r_3 = \frac{1 - n_2}{\psi_0}$$

where -1 is an arbitrary constant value selected for the second

radius. Upon this system of coördinates contour lines may be drawn connecting points for which any aberration is constant.

In Fig. 2 and Fig. 3 contour lines have been drawn, showing, respectively, values of the functions

$$\frac{VF'_{ax}-VF'_h}{f_D}$$



and

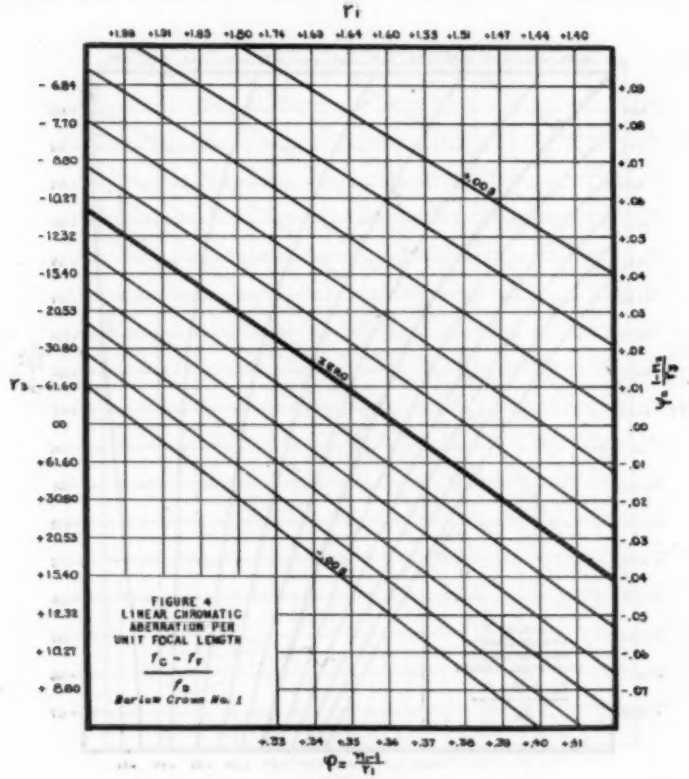
$$\frac{f_D-h/\sin\alpha'}{f_D}$$

Figs. 4 and 5 show contour lines for

$$\frac{f_{C.}-f_F.}{f_D}$$

when the barium crown numbers 1 and 2, respectively, are combined with the flint.

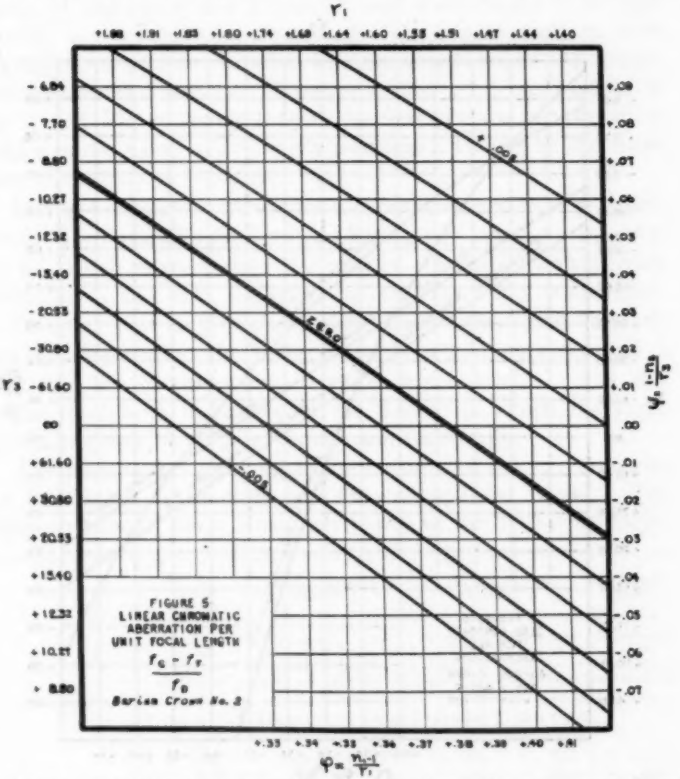
For conciseness these three functions will be referred to as the linear spherical, comatic and chromatic aberrations per unit focal length. In all the figures the values of φ and ϕ are indi-



cated along two sides of the diagram and the corresponding values of r_1 and r_3 along the two opposite sides. As previously noted, the value of r_2 for all points is -1. The values of the aberrations upon which the contours are based have been determined by trigonometric tracing of the rays. The thicknesses have not been neglected but are four and two thirty-fifths of the

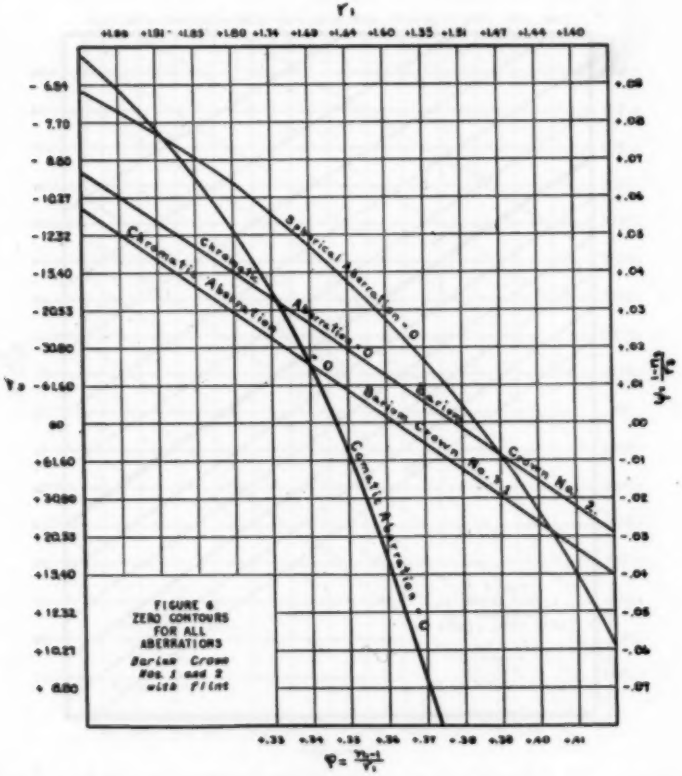
radius of curvature of the second surface for the crown and flint components, respectively. These thicknesses are large and permit the large apertures to be secured which are required for the objectives of low-power telescopes.

The uses which may be made of such a set of contours are self-



evident. If desired values of the three aberrations per unit focal length are given, it is possible to determine whether the particular glasses here selected enable the conditions to be met or the most suitable set of compromise values may be selected. The three radii may be read directly from the diagram and multiplied by a suitable factor of proportionality in order to secure the desired focal length.

It should be observed that while the contours are based upon the use of the particular glasses, the constants of which are given above, the radii may be approximately determined for any barium crown and flint for which the difference of index of refraction is .042 by use of the parameters φ and ϕ and equa-



tions 2. In such an application it is well to check by trigonometric computation the values thus obtained, after which the corrections to be applied may be determined by considering the slopes of the surfaces represented by the contours.

The tolerances which may be permitted in the manufacture of an objective can be readily determined. Consider, for exam-

ple, the objective having no spherical and no comatic aberration, $\varphi = +.3015$, $\psi = +.076$. If a variation of $\pm .0005$ from the normal value of the aberrations is allowed, an examination of the contours shows that the tolerances will not be exceeded if $\varphi = +.3015$, $\pm .003$ and $\psi = +.076 \pm .003$. Expressing this in terms of the radius of curvature, for a lens of unit focal length $r_1 = +.641 \pm .006$, $r_2 = -.336$, $r_3 = -2.72 \pm .10$. In a case such as here selected the allowable variation in focal length might well determine tolerances closer than those given above. A comparison of Fig. 4 and Fig. 5 shows that if the flint remain unchanged, a change in the value of ν of $\pm$ 0.5 produces a change in the chromatic aberration of $\pm$ 0.0005.

In Fig. 6 the zero contours for all aberrations have been superposed. Two triangles are formed (having two sides in common), one of which characterizes the results obtainable with barium crown No. 1 and flint, the other barium crown No. 2 and flint. If for either pair of glasses, all three aberrations could be made zero simultaneously, the three contours would pass through a point and the corresponding triangle would vanish. The size of the triangle is indicative of the degree to which the three corrections may be made to vanish simultaneously.

The lenses corresponding to the vertices of the two triangles are interesting as in each case two aberrations vanish. The characteristics of these lenses are given in the following table:

	Barium crown 1 and 2	Barium crown, number 1		Barium crown, number 2		Ordinary crown
	No spherical or comatic aberration	No spherical or chromatic aberration	No comatic or chromatic aberration	No spherical or chromatic aberration	No spherical or chromatic aberration	No spherical or comatic aberration
φ	+0.3015	+0.404	+0.340	+0.390	+0.3315	
ψ	+ .076	− .029	+ .015	− .009	+ .0325	
r_1	+ .641	+ .4765	+ .529	+ .501	+ .557	+ .615
r_2	− .336	− .335	− .324	− .340	− .322	− .397
r_3	−2.72	+7.13	−13.97	+23.35	−6.11	−1.44
Spherical aberration	− .0001	+ .0002	− .0035	− .0001	− .0020	− .0002
Comatic aberration	− .0000	+ .0048	− .0001	+ .0038	+ .0002	+ .0001
Chromatic aberration	[1] +.0026 [2] +.0017	.0000	− .0001	− .0001	.0000	+ .0047

[1] Barium crown No. 1.
[2] Barium crown No. 2.

The values of φ and ψ, as read from the contours and the values of the radii after multiplication by a suitable constant to give unit focal length, are given. The computations were made with five-place tables and the values of the aberrations may be in error by $\pm$ .0003. For purposes of comparison corresponding data have been given for a lens of ordinary crown (N_D 1.514, $N_D - N_C$ .00249, $N_F - N_D$ .00599, ν 60.6) combined with the flint and corrected for spherical aberration and the sine condition.

A comparison of the values of the aberrations for the different lenses is significant. As is to be expected, the barium crown having the larger value of ν, combined with the flint, is the most favorable for the elimination of all aberrations simultaneously. A consideration of the lenses corrected for spherical and comatic aberration and constructed from the two barium crowns shows that if ν were increased to approximately 59.5 all the aberrations for the aperture here considered would vanish. The advantage to be gained might, however, be illusory. The definition of chromatic aberration is, to a degree, arbitrary. If the usual curves showing the zonal aberrations for each of the colors is plotted, it will be seen that when there is a slight amount of under-correction for achromatism at the axis the correction for an outer zone may be very satisfactory. This is due to the fact that when a cemented doublet is corrected for spherical aberration for the D spectrum line, it is overcorrected for the F spectrum line. The large amount of undercorrection in the lens of ordinary crown shows the disadvantage of this type of glass as compared with the barium crown in an objective of large aperture.

Summary: A system of contours has been constructed which is helpful in the preliminary design of a telescope objective for which the desired values of the aberrations are determined.

The characteristics of some typical lenses are tabulated.

TECHNICAL STAFF,
ORDNANCE DEPARTMENT, U. S. A.
14th, July, 1920.

NOTE ON THE PATH OF A RAY THROUGH A PRISM IN A PRINCIPAL SECTION.

BY

JAMES P. C. SOUTHALL.

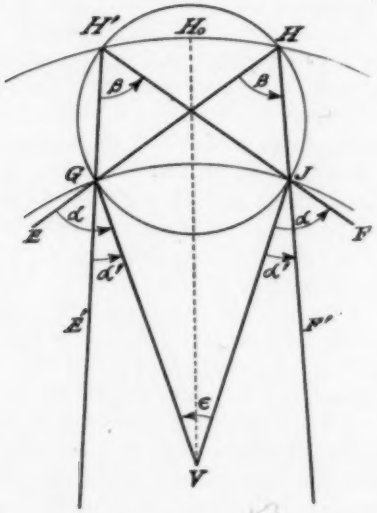
MR. T. SMITH has recently published a trigonometrical formula for the calculation of the angle of incidence of a ray which traverses a prism in a principal section and emerges under a prescribed angle of deviation.* My impression is that several years ago I saw a geometrical solution of this problem perhaps in a number of the London *Optician*, but I have not been able to obtain a file of this journal and therefore cannot verify the reference. However, the geometrical construction is quite obvious from a consideration of the accompanying diagram which shows the ordinary familiar method of tracing the path of a ray through a prism. The present problem may be stated as follows: Being given the refracting angle (β) of a prism made of glass of index of refraction n, to determine the direction (α) of an incident ray which shall traverse a principal section of the prism and emerge with a prescribed deviation (ε). Around a point V as centre describe the arcs of two concentric circles, the radius of one being n times that of the other; and construct the central angle JVG equal to the given angle of deviation (ε), so that the straight line joining the two points designated here by G and J shall be a chord of the inner circle. Describe the segment of the circle JHG which has this same chord GJ and which contains the given refracting angle (β) as inscribed angle; and let the two points of intersection of this segment with the outer of the two concentric circles be designated by H and H'. Connect H by straight lines with G and J, and let E and F designate any convenient points on the prolongations of HG and HJ beyond G and J, respectively. Then $\angle EGV = \alpha$ will be the required angle of incidence at the first face of the prism of the rav which emerges with the prescribed deviation (ε). If HE and HF are drawn normal to the two faces of the prism, then the straight lines VG and VJ will be parallel to the incident and emergent rays, respectively, and the straight line VH will be parallel to the path of the ray

*T. Smith: "The incident angles corresponding to given deviation produced by a prism." *Transactions of the Optical Society*, xxi (1919–1920), 49, 50.

inside the prism. The angle of emergence at the second face is $\angle VJF = \alpha'$.

Evidently, in general the problem has two solutions, the other one being obtained in exactly similar fashion by connecting H′ by straight lines with G and J; and in this case it is easy to see that the angle of incidence at the first face, namely α / E′GV, is equal to the angle of emergence (α') of the other ray at the second face; as indeed might easily have been inferred by reversing the

FIG. 1.



path of this latter ray. Evidently, also, the two values of α are connected by the relation:

$$\alpha + \alpha' = \beta + \epsilon.$$

Of course, in case the segment of the circle does not intersect the outer of the two concentric circles, there will be no ray which can go through the prism with the required deviation. Analytically, the conditions may be expressed as follows:

If $\tan\frac{\beta}{2} < \frac{\sin\frac{\epsilon}{2}}{n - \cos\frac{\epsilon}{2}}$, the two points designated by H, H′ will

be real and different (as shown in the figure), and the problem will have two solutions as above. If $\tan \frac{\beta}{2} = \frac{\sin \frac{\epsilon}{2}}{n - \cos \frac{\epsilon}{2}}$, the two points designated by H, H′ will coincide with each other in the point H_0 on the bisector of the angle JVG; and in this remarkable case there will be only one solution, namely, the ray which traverses the prism symmetrically so that the angles of incidence and emergence are equal ($\alpha = \alpha'$) and $\beta = 2\alpha = \epsilon_0$ where $\epsilon = \epsilon_0$ denotes now the angle of minimum deviation of the prism. And, finally, when $\tan \frac{\beta}{2} > \frac{\sin \frac{\epsilon}{2}}{n - \cos \frac{\epsilon}{2}}$, the points H, H′ are imaginary, and in this case the problem has no solution.

The trigonometrical calculation of the angle α may be derived easily from the relation:

$$\sin(\beta+\epsilon-\alpha)+\sin\alpha.\cos\beta=\sin\beta\sqrt{n^2-\sin^2\alpha},$$

which can be obtained by a consideration of the diagram. The solution of this equation is:

$$\tan\alpha=\frac{A+B}{C},$$

where

$$A=\sin(\beta+\epsilon)\left\{\cos\beta-\cos(\beta+\epsilon)\right\},$$

$$B=\sin\beta\sqrt{n^2\left\{2-n^2.\sin^2\beta-2\cos\beta.\cos(\beta+\epsilon)\right\}-\sin^2(\beta+\epsilon)},$$

$$C=(n^2-1)\sin^2\beta-\left\{\cos\beta-\cos(\beta+\epsilon)\right\}^2.$$

This formula is different from the one given by Mr. T. Smith in the paper above mentioned. It is possible that it may be arranged in a form more convenient for logarithmic computation. In case of a prism made of glass of index $n = 1.5$ whose refracting angle is $\beta = 60°$, I find, by means of this formula, the following values of the angle α corresponding to $\epsilon = 40°$:

$\alpha =$	63°	27′	27.7″
$\alpha' =$	36°	32′	32.3″
$\alpha+\alpha' =$	100°	0′	0.0″ $=\beta+\epsilon$.

DEPARTMENT OF PHYSICS,
COLUMBIA UNIVERSITY,
July 1, 1920.

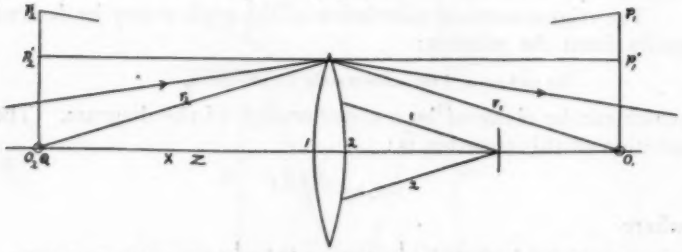
THE TREATMENT OF CERTAIN SPECIAL CASES IN LENS CALCULATION.

BY

C. W. WOODWORTH.

CERTAIN sources of error in lens calculation cannot be avoided. For example, there is no method of accurately determining a distant focus. The index of refraction of glass is usually known with great accuracy, but the remaining physical data are likely to be doubtful beyond three or four places. Assuming that all the physical data are known to five places and that the distant focus is ten thousand or upwards, it is not possible to determine

FIG. 1.



Graphic calculation of a biconvex lens; X optical axis, Z graph axis, Q graph centre, O_1O_2 centres of curvature of surfaces 1 and 2, r_1r_2 radii to points of refraction, p_1p_2 intercepts of incident and refracted ray on reference planes, p'_1 p'_2 intercepts of segment of ray within the glass, 1 and 2 lens surfaces and the corresponding graphs.

more than a single digit with certainty, while if the focus was not above one hundred, there would be four certain digits.

Fig. 1 illustrates such a case after refraction at the first surface when the ray becomes nearly parallel with the axis. The complete calculation of this ray requires nine steps and may be accomplished by the following equations:

First Surface

(1) $$\sin \alpha = \frac{o \sin \theta}{r}$$

(2) $$\sin \alpha' = \frac{\sin \alpha}{n}$$

(3) $$\theta' = \theta + \alpha' - \alpha$$

(4) $$o' = \frac{r \sin \alpha'}{\sin \theta'}$$

Second Surface

$$o' = o' + d \tag{5}$$

$$\sin \alpha' = \frac{o' \sin \theta}{r} \tag{6}$$

$$\sin \alpha = n \sin \alpha' \tag{7}$$

$$\theta = \theta + \alpha - \alpha' \tag{8}$$

$$o = \frac{r \sin \alpha}{\sin \theta} \tag{9}$$

The physical data are

radii	10
separation of surfaces	1
index of glass	1.5
focus of incident ray	28.125
slope of incident ray $\sin \theta =$	.16

Assuming all of this was known to five significant figures, the calculation would proceed.

			Sines		Arcs		Foci
First Surface		s	.16000	θ	.16068		28.125
	(1)	a	.45000	α	.46536		
	(2)	a'	.30000	α'	.30452		
		s'	.00014 (3)	θ'	.00014 (4)	o'	21428.
Second Surface	(6)	a'	.30026	α'	.30479 (5)	o'	21447.
	(7)	a	.45039	α	.46577		
		s	.16042 (8)	θ	.16112 (9)	o	28.076

In the above since θ' has only two significant figures, o' can have only one certain figure, and in fact, with a variation of less than .000005 in the value of $\sin \theta'$ the value of o' will be found to vary between 20,000 and 22,000.

The same difficulty occurs where a different method of calculation is used. The most different method is the form which the writer has recently devised for expressing the focus of a lens system as follows:

$$\frac{1}{o_m} = \frac{1}{o_1} + \sum_{k=1}^{k=m} \frac{\sin \varphi_k}{r_k} (\cot \alpha'_k - \cot \alpha_k) \tag{10}$$

and it will be evident that when $\frac{1}{o_m} = .00042$ and $\frac{1}{o_1} = .03556$, as in the above case, the quantity indicated by the summation must be negative and of almost the same magnitude; that is, .03514 giving precisely the same situation, since if these latter figures are not known beyond five places, there can be but two significant figures in the first, of which but one can be certain.

Fortunately it is rarely required to determine the magnitude of a distant focus and still more fortunate that in a calculation like that given above the uncertainty of the value of o' does not

prevent the final o value having four certain figures. This is because that while both θ' and o' are uncertain, their product is accurate.

The difference between a' of the first and second surfaces amounting to .00026 corresponds to the difference of 19 between the centres. This quantity is no more accurate than o' but the error is in the fifth digit. If this quantity was much larger, the error might reach the fourth digit.

The inaccuracy of a distant focus is a very serious matter in the case where the next surface requires a different type of equation, which is true, if it is a flat surface. Such a case occurs in the calculation of a Huygenian eyepiece, in which the extreme oblique ray may pass from the field lens to the eye lens along a course nearly parallel with the axis, and the calculation becomes and remains completely untrustworthy instead of immediately regaining its accuracy as in the preceding example.

It is necessary therefore to substitute another method of calculation for such cases and no such method has heretofore been available. A method of calculation is now proposed which entirely avoids such errors. It is based on the determination of the intercepts on a series of reference lines made by the successive segments of the ray.

The following calculation of an edge ray of violet light through the thirty-six-inch refracting telescope at the Lick Observatory illustrates the precise conformity of the proposed method with the one commonly used. The indices for this light are crown glass 1.508721, flint glass 1.618812. The remaining optical data are given in the following calculations of the distances between the centres of curvature or reference planes:

1	r =	259.52
	s =	1.96
		257.56
2	r =	259.52
	d =	517.08
	r =	259.52
	s =	8
		267.52
3	r =	239.59
	d =	27.93
	r =	239.59
	s =	.93
		240.52
4	r =	40,000.
	d =	39,759.48

		Foci	Ray Intercepts		Angles		Sines	Cosines / Tangents
1	o	∞	p	18.	α	.0694145	.0693588	
	o′	509.3270(5)	p′	11.93390	α'	.0459881	.0459719	
	d	517.08	dt′	12.11555	θ'	.0234264	.0234243	.9997256 / .0234307
2	o′	1,026.407(8)	p′	24.04945	α'	.0927765(6)	.0926435(6)	
	o	512.1781(7)	p	36.36525	α	.1402323(4)	.1397732(3)	
	d	27.93	dt	1.98306	θ	.0708822	.0708229	.9974890 / .0710011
3	o	484.2481(7)	p	34.38219	α	.1436372	.1431439	
	o′	1,342.121(12)	p′	21.18846	α'	.0885409	.0884253	
	d	39.759.48	dt′	627.69086	θ'	.0157859	.0157853	.9998754 / .0157872
4	o′	41,101.60	p′	648.87932	α'	.0162207	.0162200	
	o	40,673.37(9)	p	1,050.63428	α	.0262601	.0262571	
	u	673.4±1	u	673.4±1	θ	.0258253	.0258224	.9996666 / .0258310

In making the above calculation the ray-intercept method was first used. The equations differ from those used above as follows:

Instead of (1) and (6) use

$$\sin\alpha = \frac{p\cos\theta}{r} \tag{11}$$

instead of (4) and (9) use

$$p = r\frac{\sin\alpha}{\cos\theta} \tag{12}$$

and instead of (5) use

$$p = p + dt \tag{13}$$

and finally determine the focus by

$$o = p/\text{tang}\,\theta \tag{14}$$

and

$$u = r - o \tag{15}$$

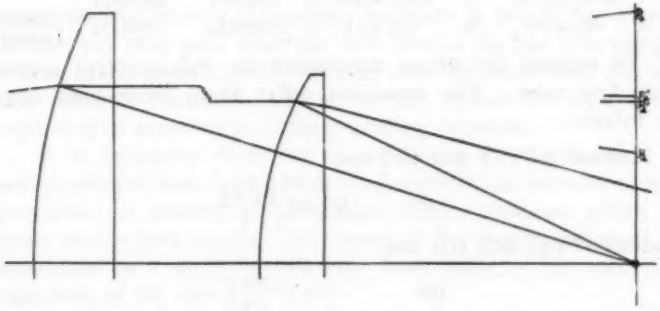
This last equation applying to both methods.

After calculating the above (but not including column one) according to the intercept method, the lens was then recalculated by the former equations and all angular differences indicated by giving in parentheses the final digit when different. In no case was the difference greater than .0000001. This recalculation involves the calculation of the foci given in column one. After obtaining these figures the foci were then recalculated from the intercepts by equation (14), and as before the differences indicated in parentheses and the variation in each case is not larger than a figure in the seventh place.

The intercept method thus affords a most excellent checking method against the focus method of calculation. It is more than this, because it will detect and avoid the error referred to above, such as may occur, for instance, with what I have called the comatic ray in an eyepiece calculation, as shown in Fig. 2.

It will only be necessary in illustrating its superiority in such a case to trace the ray from the time it leaves surface two till it reaches surface four, calculating the ray from the eye downward

FIG. 2.



Lenses of a Huygenian eyepiece so drawn that the two curved surfaces have a common centre. p_2 intercept of the ray in the eye lens, p'_3 intercept of the ray passing from eye lens to field lens, p'_4 intercept of the ray in the field lens, and p_4 intercept of the ray below the field lens.

through the lenses. The error showing in the value found for $\sin \alpha_4$.

The intercept p_2 is 6.6 mm. The radius of surface two is 15 mm., of surface four 24 mm. The glass index for both lenses is 1.5085. The value of the angle θ is .00500 corresponding with $\sin \alpha_2$ .44. The distance from the reference line of surface two to surface three is 46.8 mm. From surface three to the reference line of surface four is 20.8 mm.

The calculation by the focus method proceeds

o_2	1320.0		
d_2	46.8		
o_3	1366.8	θ_2	.00500
o'_3	2061.8	θ_4	.00331
d_3	20.8		
o'_4	2041.0	$\sin \alpha'_4$	$.28148 = 40$

By the intercept method we have

p_2 6.6 θ_3 .00500
p_3 6.834 θ_4 .00331 cos .99999
p_4 6.7652 sin α'_4 .28188 ± 1

The angle θ in the preceding calculations is so small that arc sine and tangent are equal for more than the number of decimal places considered and therefore equation (2) becomes $\theta_4 = \theta_3/n$. In both calculations the plane surface three is used as a reference plane, since evidently the centre of curvature at infinity cannot be employed.

These two calculations show a difference of four in the fourth place, the latter being the correct value of this sine. A difference of .000005 in θ_3 would make this same amount of difference in the fourth place by the focus method and only one in the fifth place by the intercept method as indicated by the ± signs. The latter method is thus shown to be about forty times more accurate than the former method in this case where the θ angle is .005. If the angle were still smaller, the difference of accuracy would very rapidly increase.

Another source of error is presented in the calculation given above of the Lick Observatory telescope. This error is due to the very great radius length of the fourth surface. The calculation given above indicated an error in the fourth place, even though the seven-place functions were employed. The error is actually twice as great as here indicated, as will be shown below.

The following method enables one to entirely avoid this source of error increasing the accuracy of the final result by two digits. The plan is to modify the intercept method by substituting at the last refraction a plane tangent to the vertex of the lens in place of the reference plane through the centre of curvature. This requires the development of a new equation. In Fig. 3 one will see that

$$(16) \quad h''+h'+p'=r\tan(\alpha'-\theta')$$

and also that

$$(17) \quad h''=h'\frac{\tan(\alpha'-\theta')}{\tan\theta'}$$

Substitute this value in equation (16) and we may write the combined equation in the form

$$(18) \quad h'=\frac{r\tan(\alpha-\theta')-p}{1-\frac{\tan(\alpha'-\theta')}{\tan\theta'}}$$

and since, as likewise shown in the figure,

$$(19) \qquad h = h' \frac{\tan \theta}{\tan \theta'}$$

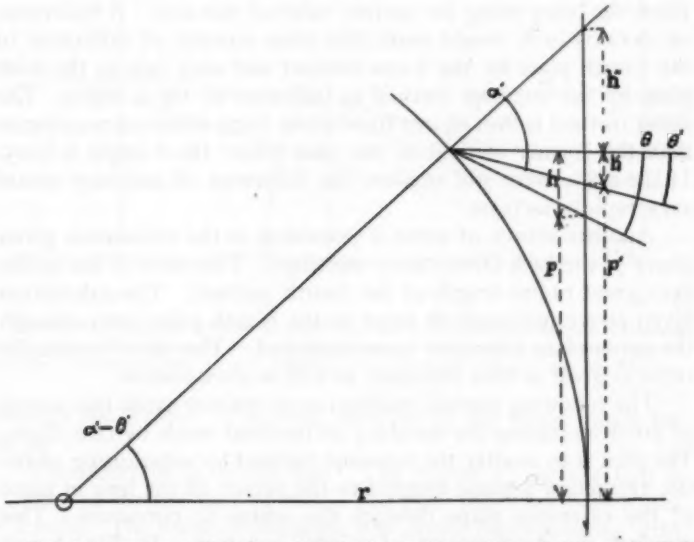
and

$$(20) \qquad p = p' + h' - h$$

we may combine the last three equations and write the final equation in the form

$$(21) \qquad p = p' - \frac{\tan \theta - \tan \theta'}{\tan \theta' - \tan (\alpha' - \theta')} \left\{ r \tan (\alpha' - \theta') - p \right\}$$

FIG. 3.



Ray intercepts on a plane tangent to the lens vertex α, α', θ and θ' the angles made by the incident and refracted rays with the radius at the point of refraction and with the optical axis, p p' the intercepts on the vertex plane of the incident and refracted rays, h, h' and h'' the difference between the height of the incident point from the optical axis and these intercepts and that of the radius.

In order to apply this equation to the calculation of the fourth surface, we must substitute a different value of d_3, because of the new position of the reference plane. This new value is 240.52, and therefore

p'_3	21.18846
dt	3.79714
p'_4	17.39132

Stated numerically, equation (21) is

$$p = 17.39132 \frac{-.025809 - .0157872}{.0157872 - .0004348}(40000 \times .0004348 - 17.39132)$$
$$= 17.39132 \frac{.0100438}{.005743} .00065$$
$$= 17.39132 - .00114$$
$$= 17.39018$$

for which the value of u is 673.229 ± 1 instead of 673.4, as previously determined.

The determination of the amount of the error in this and in the former calculation is obtained by augmenting and decreasing the last value of tan θ (or of sin θ) .00000005. and represents only the error in the last operation.

In all the above calculations angles are given in radians and the natural functions are employed, the arcs and functions being determined by the use of the well-known trigonometric series.

The intercept method of lens calculations applies to three conditions, the ordinary one of a curved surface with short radii, the case of a flat surface with an infinite radius, and also the intermediate case of the lens of slight curvature and long radius, and it permits the use of a lesser number of decimal places since the practice of using a large number is in order to secure sufficiently accurate results, notwithstanding such sources of errors as are pointed out above. The new method can be generally used as a substitute for all prevailing methods, including paraxial rays as well as rays affected by spherical aberrations. It is as satisfactory in all cases, is more accurate in some cases, as shown above, and has the additional advantage of corresponding term by term with the steps of the author's graphic methods of lens calculation.

UNIVERSITY OF CALIFORNIA.

NOTE ON CALCULATION OF PATH OF RAY THROUGH A SYMMETRICAL OPTICAL INSTRUMENT.

BY

JAMES P. C. SOUTHALL.

IF the radius (r) of a spherical refracting surface is measured along the optical axis from the vertex A to the centre C, that is, if $r = AC$, the refracting power will be

$$F = (n' - n)R, \tag{1}$$

where $R = 1/r$ denotes the curvature of the surface and n, n′ denote the indices of refraction of the two media in the order in which the light traverses them.

Let α, α' denote the angles of incidence, and refraction of a ray of light which is refracted at this surface, and which crosses the straight line AC before and after refraction at points designated by L and L′ and at slopes denoted by θ and θ', respectively; so that (taking account of the signs of these angles) we have the well-known invariant relation:

$$\alpha' - \theta' = \alpha - \theta. \tag{2}$$

And, finally, let the abscissæ of the points L, L′ measured along the optical axis from the centre C as origin be denoted by c, c' ($c = CL$, $c' = CL'$); and put $C = n/c$, $C' = n'/c'$. With this notation we derive immediately the following geometrical relations:

$$\frac{C \cdot \sin \alpha}{n \cdot \sin \theta} = \frac{C' \cdot \sin \alpha'}{n' \cdot \sin \theta'} = -R; \tag{3}$$

and since the ray is subject to the optical law of refraction, *viz.*,

$$n' \cdot \sin \alpha' = n \cdot \sin \alpha, \tag{4}$$

we find by imposing this condition that the ray-coördinates are connected with each other by the following equation:

$$\frac{n'^2}{C'} \sin \theta' = \frac{n^2}{C} \sin \theta. \tag{5}$$

So far there is nothing at all novel about these formulæ except perhaps the mode of writing them. Now let us introduce by way of abbreviation the symbol E to denote a function of the angles α, α', θ and θ' which is defined as follows:

$$E = (\sin \theta' - \sin \alpha') - (\sin \theta - \sin \alpha). \tag{6}$$

Evidently with the aid of (1), (3) and (4) we can write:

$$\frac{\sin\theta'}{\sin\theta} = 1 + \frac{E}{\sin\theta} + \frac{n}{n'} \cdot \frac{F}{C},$$

so that from (5) we derive the formula:

$$\frac{n}{n'} C' = \frac{n'}{n} C\left(1 + \frac{E}{\sin\theta}\right) + F; \tag{7}$$

which may be written also as follows:

$$\frac{C'}{n'^2} = \frac{C}{n^2} + \frac{F}{nn'} - \frac{R}{n \cdot \sin\alpha} E. \tag{7a}$$

The function E may be put in a form more convenient for logarithmic computation; for, since according to (2),

$$\alpha + \alpha' + \theta + \theta' = 2(\alpha' + \theta), \quad \alpha - \alpha' + \theta - \theta' = 2(\alpha - \alpha'),$$

obviously,

$$E = 4 \sin\frac{\alpha - \theta}{2} \cdot \sin\frac{\alpha' + \theta}{2} \cdot \sin\frac{\alpha' - \alpha}{2}. \tag{8}$$

Obviously, also, in the case of a paraxial ray for which all the angles are so small that their second and higher powers may be neglected, this function E vanishes entirely; and if the symbols B, B' are employed instead of C, C' to indicate that the ray is a paraxial ray, formula (7) becomes:

$$\frac{n}{n'} B' = \frac{n'}{n} B + F. \tag{9}$$

The trigonometrical expressions which occur in the formulæ (3) and (5) suggest immediately the central perpendiculars let fall on the incident ray and the corresponding refracted ray which have been frequently employed in systems of formulæ for lens-calculation as well as the perpendiculars let fall from the vertex of the surface. For if p, p' denote the lengths of the perpendiculars dropped from the centre C on the ray before and after refraction, respectively, then

$$p = \frac{\sin\alpha}{R} = -\frac{n \cdot \sin\theta}{C}, \; p' = \frac{\sin\alpha'}{R} = -\frac{n' \cdot \sin\theta'}{C'}, \; n' \cdot p' = n \cdot p. \tag{10}$$

Thus, formula (7a) may be expressed in terms of p as follows:

$$\frac{C'}{n'^2} = \frac{C}{n^2} + \frac{F}{n \cdot n'} - \frac{E}{n \cdot p}.$$

And therefore for an edge-ray and a paraxial ray refracted at the

kth surface of a centred system of spherical refracting surfaces we have the following recurrent formulæ:

$$\left.\begin{aligned} \frac{C'_k}{n_k^2+1} &= \frac{C_k}{n_k^2} + \frac{F_k}{n_k \cdot n_k+1} - \frac{E_k}{n_k \cdot p_k}, \\ \frac{B'_k}{n_k^2+1} &= \frac{B_k}{n_k^2} + \frac{F_k}{n_k \cdot n_k+1}. \end{aligned}\right\} \quad (11)$$

Now if these two equations are subtracted from each other and the result cleared of fractions, taking account of the relation $n_k . p_k = n_{k+1} . p_k'$, and if, finally, we put

$$C-B=\delta C, \quad C'-B'=\delta C',$$

we shall obtain the result:

$$p_k'^2 . \delta C_k' = p_k^2 . \delta C_k - n_k . p_k . E_k ; \quad (12)$$

which enables us to find the spherical aberration after refraction at the kth surface in case we know its value before refraction.* In passing to the next succeeding surface we may write:

$$\delta C_{k+1} = \left(\frac{C_{k+1}}{C_k'}\right)^2 \delta C_k', \text{ approximately;}$$

and since it follows from (10) that

$$p_{k+1} . C_{k+1} = p_k' . C_k', \quad (13)$$

the last result is equivalent to the equation:

$$p_{k+1}^2 . \delta C_{k+1} = p_k'^2 . \delta C_k', \text{ approximately.} \quad (14)$$

Consequently, if we give k in succession all integral values from $k=1$ to $k=m$, where m denotes the total number of refracting surfaces, the recurrent formulæ (12) and (14) taken together

* It may be of interest to see how this formula (12) may be derived very simply starting from the abscissa-equation for a paraxial ray as given by formula (9). With the aid of (1) this latter formula may be put in an invariant form as follows:

$$\frac{1}{n'}\left(\frac{B'}{n'}+R\right) = \frac{1}{n}\left(\frac{B}{n}+R\right).$$

Now according to (10):

$$\frac{B}{n}+R = -\frac{\delta C}{n} + \frac{C}{n}+R = -\frac{\delta C}{n} + \frac{\sin a - \sin \theta}{\rho};$$

and since for the refracted ray a precisely similar expression may be obtained by priming all the letters except R, the following equation may be written immediately:

$$\frac{\delta C'}{n'^2} - \frac{\sin a' - \sin \theta'}{n' \rho'} = \frac{\delta C}{n^2} - \frac{\sin a - \sin \theta}{n \rho}.$$

which is obviously capable of being transformed into (12).

yield the following equation for calculating the spherical aberration:

$$p_{m'^2} \cdot \delta C_{m'} = p_1^2 \cdot \delta C_1 - \sum_{k=1}^{k=m} n_k \cdot p_k \cdot E_k . \tag{15}$$

Now if the incident paraxial ray and marginal ray cross the axis at the same point L_1, we shall have $\delta C_1 = 0$, as is always the case for a monocentric bundle of incident rays emanating from a point on the optical axis. Moreover, if the points where the paraxial ray and the edge-ray cross the optical axis in the last medium are designated by M_{m+1} and L_{m+1}, respectively, then the linear measure of the spherical aberration along the axis will be $\delta c_{m'} = M_{m+1} L_{m+1}$; and since

$$\delta c'_m = -n_{m+1} \frac{\delta C_{m'}}{C_{m'^2}}, \text{ approximately,} \tag{16}$$

we obtain finally the following formula for calculating the spherical aberration:

$$\delta c_{m'} = \frac{n_{m+1}}{(p_{m'} \cdot C_{m'})^2} \sum_{k=1}^{k=m} n_k \cdot p_k \cdot E_k, \quad (\delta c_1 = 0). \tag{17}$$

Summing up these results, we have the following system of formulæ:

$$\left.\begin{aligned} & p_k = -\frac{n_k \cdot \sin \theta_k}{C_k}, \quad p_k' = \frac{n_k}{n_{k+1}} p_k, \\ & \sin \alpha_k = p_k \cdot R_k, \quad \sin \alpha_k' = p_k' \cdot R_k, \\ & \theta_{k+1} = \theta_k + \alpha_k' - \alpha_k, \\ & C_k' = -\frac{n_{k+1} \cdot \sin \theta_{k+1}}{p_k'}, \\ & n_k \cdot p_k \cdot E_k = (n_k \cdot p_k)^2 \left(\frac{F_k}{n_k \cdot n_{k+1}} + \frac{C_k}{n_k^2} - \frac{C_k'}{n_k^2 + 1} \right). \end{aligned}\right\} \tag{18}$$

And since

$$a_k = d_k + r_{k+1} - r_k, \tag{19}$$

where $d_k = A_k A_{k+1}$ and $a_k = C_k C_{k+1}$, we pass from the kth to the $(k+1)$th surface by means of the formula:

$$C_{k+1} = \frac{n_{k+1} \cdot C_k'}{n_{k+1} - a_k \cdot C_k'}. \tag{20}$$

Or, without computing the values of C and C', instead of (20), we may employ the following formula:

$$p_{k+1} = p_k' + a_k \cdot \sin \theta_{k+1}, \tag{21}$$

and calculate E_k by means of either of the two following expres-

sions, the first one being convenient for machine calculation, and the second being adapted to logarithmic work:

$$E_k = \sin\theta_{k+1} - \sin\alpha_k' + \sin\alpha_k - \sin\theta_k; \tag{22}$$

$$E_k = 4\sin\frac{\alpha_k - \theta_k}{2}\sin\frac{\alpha_k' + \theta_k}{2}\sin\frac{\alpha_k' - \alpha_k}{2}. \tag{23}$$

As a numerical illustration we may compute the object-glass of a telescope composed of a triple cemented lens of two kinds of glass, crown and flint, with following data:

	k=1	k=2	k=3	k=4
Radii: r_k	53·4759	34·6075	−41·7573	−171·2457
Thicknesses: d_k	0.4994	0.9988	0.4994	

Indices of refraction (for Fraunhofer D-line): $n_1 = n_5 = 1$ (air),

$n_2 = n_4 = 1.61358$ (flint), $n_3 = 1.51806$ (crown).

The so-called edge-ray is supposed to be initially parallel to the axis ($\theta_1 = 0$, $C_1 = 0$) at a distance from it equal to 6 ($p_1 = 6$). The focal length of the lens will be found to be 100.

The following values were obtained by logarithmic computation according to formulæ (18), (19) and (20):

	k=1	k=2	k=3	k=4
lg F_k	8.0597131	7.4409235−	7.3593612−	7.5542512
lg C_k		8.1858497	7.8511258	7.5849386
lg C_k'	8.2692065	8.0398055	7.7444548	7.5684442
lg p_k	0.7781513	0.6537176	0.8688988	1.0019135
lg $n_k.p_k.E_k$	7.0024979−	6.6982407	7.2368742	7.0815801−

(A minus sign appended to a logarithm indicates that the number itself is negative.) Thus we find by formula (17) that

$\delta c_4' = 0.0034$, approximately.

Following is same calculation made on a machine which will be found to be considerably quicker. The C's do not have to be calculated, but only the p's. We use here the formulæ in (18) which do not involve C or C', formula (21) and formula (22). In formula (17) we put $\sin\theta_{m+1} = p_m'.C_m'$.

	Angles	Sines	Central perpendiculars
θ_1	0° 0′ 0.00″	0.0000000	p_1 = 6.00000
α_1	6° 26′ 31.74″	0.1122000	
α_1'	3° 59′ 14.16″	0.0695348	
θ_2	− 2° 27′ 17.58″	−0.0428327	p_1' = 3.71844
α_2	7° 28′ 48.21″	0.1301811	p_2 = 4.50523
α_2'	7° 57′ 13.17″	0.1383721	
θ_3	− 1° 58′ 52.62″	−0.0345730	p_2' = 4.78871
α_3	−10° 11′ 58.76″	−0.1770788	p_3 = 7.39434
α_3'	− 9° 35′ 23.93″	−0.1665963	
θ_4	− 1° 22′ 17.79″	−0.0239368	p_3' = 6.95662
α_4	− 3° 21′ 45.21″	−0.0586540	p_4 = 10.04413
α_4'	− 5° 25′ 50.77″	−0.0946429	
θ_5	− 3° 26′ 23.35″	−0.0600001	

Here we find:

	k=1	k=2	k=3	k=4
E_k	−0.0001675	0.0000687	0.0001537	−0.0000744
$n_k p_k E_k$	−0.00100500	0.00049942	0.00172529	−0.00120580

and hence $\Sigma\, n p E = 0.00001391$, which divided by $\sin^2\theta_5$ gives $c_4' = F'L_5 = 0.003863$, approximately. (Here F′ designates the focal point of the object-glass.)

Exact calculation of the two rays (paraxial-ray and edge-ray) with a calculating machine gives for the edge-ray:

$$C_4 L_5 = 270.1185,$$

and for the paraxial-ray:

$$C_4 F' = 270.1167;$$

so that the correct value of the spherical aberration is:

$$\delta c_4' = 0.0018,$$

perhaps as accurately as it can be computed. And while the approximate result is seen to be about twice as great as this value, nevertheless the two results are in substantial agreement, at least, so far as the position of the first significant figure is concerned, and both show that the system is practically free from spherical aberration. The approximate method here is at least as good and reliable as any other approximate method that has been devised. But the chief interest in the whole matter is rather theoretical than practical. After all it is nearly as easy to calculate the paraxial-ray, and indeed this calculation generally has to be performed anyhow.

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A NEW METHOD FOR THE MEASUREMENT OF PHOTOGRAPHIC FILTER FACTORS.*

BY

RAYMOND DAVIS.

ORTHOCHROMATIC and panchromatic plates are almost always used with photographic filters (colored screens of glass or gelatine) to limit the color and intensity of the light falling on the photographic plates. Since the use of a filter decreases the intensity of certain colors on the plate, the photographic exposure through a filter will be greater than that required for the same plate used without a filter.

A filter factor is the ratio of the exposure required with a given filter to the exposure without the filter; hence, to obtain the proper time of exposure when using the filter, the time of exposure without the filter should be multiplied by the filter factor.

Heretofore, so far as the author knows, filter factors have been determined by either of two methods. First, the method of trial and error, and second, the measurement of plate speeds without and with the given filter in some form of sensitometer. In the course of a study of the plates and films made in the United States, it was desired to measure the filter factors of all orthochromatic and panchromatic emulsions for a number of different filters.

A device occurred to the author for measuring the factors with the same accuracy as a careful speed measurement and yet with very much less effort.

The apparatus is constructed as follows: Light from the two sides of a standard metal filament lamp L, is reflected by similar reflectors M_1, M_2, M_3, M_4, and by similar prisms, P_1 and P_2, so that the two beams fall side by side on the photographic plate E, the filter factors of which are to be determined. In front of the plate is a simple shutter. The source of light L is movable along the line joining M_1 and M_3. At F, between M_3 and M_4, carriers to hold the filters employed may be inserted.

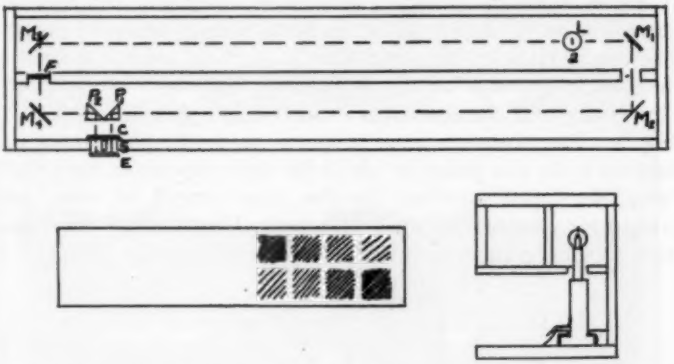
The method of procedure is as follows: The lamp initially

* Published by permission of Director, Bureau of Standards.

at the position *a*, will give two beams of equal intensity at a photographic plate, if the distances traversed by the two beams are the same and if identical optical conditions of reflection and absorption obtain for the various media through which the beams pass. If a filter is inserted in one path, the intensity of that beam will be decreased, and in order to make the beams of the same intensity it will be necessary to shift the lamp toward the filter, thus shortening that path and increasing the other. The ratio by which the filter decreases the intensity of the light is equal to the square of the length of the path which contains the filter divided by the square of the length of the path without the filter.

In practice the point of balance is obtained by making a series

FIG. 1.



Schematic arrangement of apparatus. Upper: Plan of filter factor apparatus. Left: Specimen of test plate. Right: Vertical section.

of exposures on the plate with the lamp at arbitrary distances from the centre position *a*, choosing these distances so that some are on one side of the point of balance and some on the other. The plate is developed and fixed and the density differences measured on a polarization photometer. These differences are plotted as positive or negative according to whether the setting made the lamp distance too large or too small, and the setting for a balance is found by interpolation. The distance of the new point of balance from the position of equilibrium without a filter, together with the distance from the lamp to the plate by both paths in

its position of equilibrium without a filter, enable the law of inverse squares to be applied and the filter factors calculated.

The apparatus as constructed has each path about one metre long when the lamp is in the zero position, and the lamp has a movement of about 69 centimetres so that the range of factors which can be measured is from one to thirty. The filters used must be larger than ½ inch square. The strips of the photographic plate tested are 1⅛ inches x 5 inches and two filter factors may be measured conveniently on each. The exposure time is about half a second, since the intensity of the lamp used with its compensating filter is 2¾ C.P.

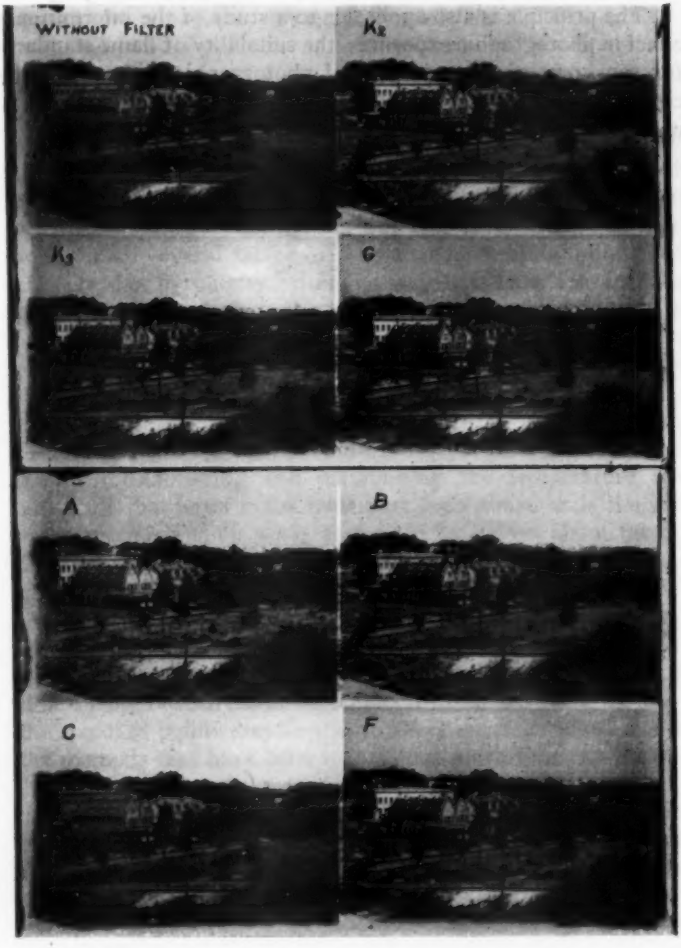
In order to get filter factors which can be used out of doors, it is necessary to modify the distribution of spectral energy of the metal filament lamp by placing in both beams a suitable screen. It is also necessary to burn the lamp at a constant and specified current so as to keep constant its spectral distribution of energy. The combination of lamp and filter is identical with that used as the standard source of the Bureau of Standards sensitometer. How successful the apparatus described has proven is shown in Fig. 2. The exposures were proportional to the filter factors as measured, the two plates on which the eight exposures were made being developed together for the same length of time, and printed together on one sheet of paper. The Wratten filters and their factors on the plate used (Wratten M) were as follows:

Without filter		1
K_1	"	4
K_3	"	4.5
G	"	5.5
A	"	18
B	"	14
C	"	12
F	"	24

In conclusion, attention may be called to other possible applications of the principle of this apparatus, which is essentially a photographic photometer.

If the plate-holder is made moveable, between two light sources, it would be easy by means of narrow banded filters, to measure the relative candle power of two sources for any part of the spectrum; to color match two lamps through variation of the current in one or both of them; to compare electric standards with flame standards, to study the effect of humidity and atmos-

FIG. 2.



Photograph to test filter factors.

pheric pressure since this effect is probably greater in the blue, and the photographic plate therefore better adapted than the eye for observing it.

The principle is also applicable to a study of the intermittency effect in photographic exposures, the suitability of flame standards for sensitometry, comparisons of photographic with visual efficiency and other problems of a similar character.

WASHINGTON, D. C.
July, 29, 1920.

1919 REPORT OF STANDARDS COMMITTEE ON PYROMETRY.

BY

W. E. FORSYTHE, Chairman.

In the report of the Standards Committee for 1919, Doctor Nutting called attention to the need of collecting reference material in such a form that it would be useful to as many persons as possible. Following out that suggestion, this report will contain several tables of data that have been found useful in the field of pyrometry. Some of these data have been computed and some of them have been taken from various papers that have already been published. They all depend upon the values of the radiation constants included in this paper, which best represent all the experimental facts.

In the past when an experimenter desired to do some work in the region of high temperatures, a great part of his time was spent in establishing and maintaining the temperature scale. The scale that has for the most part been aimed at is the centigrade thermodynamic scale, as defined by Kelvin about the middle of the last century. There has been considerable work done in the field of radiation and pyrometry in the last few years, so that now methods and constants are available that had to be worked out by each experimenter in the past. Even at the present time the experimenter in this region must spend considerable time in the measurement of temperature, but by making use of the standard points that have been very well determined and also the methods that have been worked out much time can be saved.

The summary of the data included in this report are:

1. The temperature scale and fixed points, including methods for establishing and maintaining the high temperature scale, and recommended methods for calibrating and intercomparing technical pyrometers.
2. Brightness of black body at various temperatures and related data.
3. Spectral energies of the black body at various temperatures.
4. The constants of the black-body laws.
5. The emissive powers of various metals.

TEMPERATURE SCALE AND FIXED POINTS.

One of the most important single items in pyrometry and radiation is the temperature scale. For this reason the first table of data is a summary of the more important fixed points. The temperature scale is defined in terms of the fundamental interval, *i.e.*, the interval of temperature from melting ice to boiling water under standard conditions. On the standard scale the interval is divided up into 100 degrees. The magnitude of the degree will in general depend upon the method used in measuring the temperature in this interval. It has therefore been decided to define the temperature in terms of a constant-volume hydrogen thermometer under specified conditions.

The standard scale that is aimed at for all temperatures is Kelvin's thermodynamic scale which does not depend upon the properties of any substance. This is the temperature scale that would be obtained with a gas thermometer using a perfect gas. The temperature scale obtained by a real gas can be corrected to the thermodynamic scale if the departure of the gas from a perfect gas is known. A very good discussion of this is given by Buckingham in the Bulletin of the Bureau of Standards, vol. 3, page 237. In his paper it is shown that the value that best fits all the experimental data for the ice point on the thermodynamic scale is 273.13° K. The corrections to the different gas thermometers are also given. These corrections have been worked out for temperatures up to 1000° C., and by extrapolation they have been extended to 2000° K., and are given in Table I. These corrections were calculated for an initial pressure

TABLE I.

Corrections to Constant-Volume and Constant-Pressure Scale for Nitrogen (All Corrections to be Added Algebraically. Pressure = 1000 mm. Hg.).

Temperature in degrees K.	Constant volume	Constant pressure
173*	−0.06	+0.34
273	0	0.0
600	+0.10	.30
800	.24	.67
1000	.42	1.08
1200	.64	1.54
1400	.86	1.98
1600*	1.1	2.5
1800*	1.4	3.0
2000*	1.7	3.5

* Extrapolated values.

on the gas of 1000 mm. of mercury. It was shown by Buckingham that "the thermodynamic correction of the centigrade constant-pressure scale at the given temperature is very nearly proportional to the constant pressure at which the gas is kept" and "that the thermodynamic correction of the centigrade constant-volume scale is approximately proportional to the initial pressure at the ice point."

These two propositions are very useful, since after finding the corrections for any pressure for a gas thermometer of either type, it is possible to calculate very approximately the corrections for the same gas at any other pressure.

The region of the gas thermometer is limited by the container for the gas. Day and Sosman [1] carried the gas thermometer up to the melting-point of palladium several years ago and to date there has been indication of but a very small error at this high point, *i.e.*, an error of about twice what they set as their limit of error. However, for most work, the region of the gas thermometer would be considered to end at about 1000° C. (or 1273° K.).[2] Above this, temperature is measured by means of one of the radiation laws. In whatever manner the temperature is measured in practice, it is the thermodynamic scale that is aimed at.

In order to secure a uniform temperature scale there has been a proposed international agreement by representatives of the various standardizing laboratories in which standard values and methods have been agreed upon. According to this proposed international agreement the temperature scale has been divided up into intervals according to the method used in measuring the temperature in the different intervals. The first interval is from −40° C. to 450° C. and the temperature is to be measured in this interval by means of a platinum resistance thermometer calibrated at the temperature of melting ice (0° C.), the temperature of saturated steam (100° C.) and the temperature of sulphur vapor (444.6° C.), all under standard atmospheric pressure. Points on the temperature scale between the fixed points are found by interpolation, using the relation between the resis-

[1] *Am. Jour. Sci.*, xxix, p. 93, 1910.

[2] In many laboratories temperatures are all given on the absolute centigrade scale and marked °K (Kelvin), that is they are given as degrees centigrade +273.1°.

tance and temperature expressed by one of the Callender formulæ.

$$(1)\ Pt = \frac{Rt - R_0}{R_{100} - R_0} 100, \qquad (2)\ t - Pt = \delta\left(\frac{t}{100} - 1\right)\frac{t}{100},$$

where t is the temperature, R the resistance, and δ a constant, Pt the platinum temperature.

Other fixed points in this interval that have been found useful in calibration are given in the first part of Table II.

The second interval agreed upon extends from 450° C. to 1100° C. The temperature in this interval is generally defined as measured by a standard platinum platinum-rhodium thermocouple that has been calibrated at certain standard fixed points. The thermocouple is generally calibrated at the freezing points of zinc (or cadmium), antimony and copper. The relation between E.M.F. of the thermocouple and temperature of the fixed points that is then used to calculate intermediate points is expressed by the following equation:

$$(3) \qquad e = a + bt + ct^2$$

the constants being obtained from the calibrations at the three fixed points. This connects the scales thus defined for the two intervals so that the temperature scale is continuous and still there is no uncertainty due to the use of different methods to measure the standard temperature in the same region. Other standard fixed points that are useful in calibration are given in the second part of Table II. The values given in Table II are supposed to be on the thermodynamic scale and are adopted as the values that best represent all the experimental data. In calibrating a platinum platinum-rhodium thermocouple after finding the E.M.F. of the fixed points, there is some difficulty in the interpolation for different points between two fixed points. Sosman [3] has outlined a very good method for doing this. The E.M.F. of a standard thermocouple at the fixed points has been experimentally determined. Now, if the differences between these standard values and the values found for the couple being calibrated are obtained and these differences plotted against the temperatures of the fixed points, a curve will be obtained from which the E.M.F. of any temperature can be obtained, or from which the temperature for any E.M.F. can be readily found.

No attempt has been made as yet to fix upon an international

[3] Sosman, *Am. Jour. of Sci.*, XXX, 1, 1910.

standard scale above 1100° C. (or 1373° K.). However, most experimenters that work in this region base their results upon certain of the radiation laws with particular values of the constants of these laws and certain fixed points. The radiation laws all apply to the black body and thus the temperature of a non-black body cannot be directly determined from the measurements of the energy radiated. When it is necessary to measure the temperature of non-black bodies from the intensity of the energy radiated, correction must be made for the emissive power. This correction for the emissive power, together with an explanation of this term, is more fully discussed below.

The melting point of gold, which is now quite generally accepted as 1336° K., is a very convenient standard fixed point for the calibration of pyrometers that are based upon the radiation laws. Another satisfactory point for the calibration of pyrometers is the palladium point. The value of this point is somewhat in doubt. Day and Sosman, in their extended work with the nitrogen thermometer, found 1823° K. as the value of this melting point. According to some recent work[4] this value seems too low if the value 14,350 is used for the constant c_2 of Wien's equation. It seems, therefore, that it should be raised about 5° C. or 1828° K.

The temperature scale in the region above 1336° K. is generally determined by means of an optical pyrometer with a calibration that is based upon Wien's equation.

$$J_\lambda = c_1 \lambda^{-5} e^{-\frac{c_2}{\lambda T}} \tag{4}$$

By comparing the brightnesses of a black body at two temperatures and applying this equation, the following formula is obtained:

$$\log R = \frac{c_2 \log e}{\lambda}\left(\frac{1}{T_2} - \frac{1}{T_1}\right) \tag{5}$$

where R is the ratio of the brightnesses, λ the wave-length used, T_1 and T_2 the two temperatures, and $c_2 = 14{,}350\mu$ deg. ($\mu = 0.001$ mm.). Thus, if R is measured and one temperature known, the other can be calculated.

Some of the better-known temperatures in this region are given in the third part of Table II.

[4] *Astrophys. Jour.*, li, 244, 1920. Hoffmann und Meissner, *Ann. d. Phys.* 60, 201, 1919.

ESTABLISHING THE TEMPERATURE SCALE.

The temperature of the fixed points below 1100° C. are now pretty well known. As was pointed out above, certain values have been decided upon by agreement for the standard fixed points below 1100° C. These, of course, represent the best experimental data available and are probably as accurate as can be obtained without a very great amount of additional work. Until better are available therefore, the values given in this part

TABLE II.

Standard Fixed Points.

	Degrees.
PART 1.—Interval—40° C. to 450° C.	
Freezing point of mercury	−38.88° C.
Freezing point of tin	231.84
Freezing point of cadmium	320.9
Freezing point of zinc	419.4
Boiling point of naphthalene	217.96 +0.058 (p—760)
Boiling point of benzophenone	305.9 +0.063 (p—760)
Boiling point of sulphur	444.6
PART 2.—Interval 450° C. to 1100° C.	
Freezing point of antimony	630. ° C.
Freezing point of aluminum	658.7
Freezing point of silver	960.5
Freezing point of sodium chloride	801.
Freezing point of gold	1063.
Freezing point of copper, free from oxide	1083.
PART 3.—Interval above 1100° C. (1373° K.)	
Melting point of palladium	1828° K.
Melting point of platinum	2034
Melting point of molybdenum	2810
Melting point of tungsten	3675

of Table II should be used. The values in the region above this represent the best available data and are now very well known. After the temperatures of the fixed points have been determined by one of the standard methods, *i.e.*, in terms of the gas thermometer, corrected to the thermodynamic scale, in the gas thermometer region, or by the optical pyrometer or total radiation pyrometer in the region above this, there comes the difficulty of maintaining this scale both from the standpoint of the standardizing laboratory and the man using the temperature scale in the practical field. In the region of the platinum-resistance thermometer and the thermocouple, standards of either can be obtained from the standardizing laboratories and these used for checking up the regular instruments. It is not very difficult to actually check up a resistance thermometer at any one of the standard points in the region −40° C. to 450° C. It is a little more difficult to check

up the thermocouple in the region 450° C. to 1100° C. Most of the standard fixed points in this region are given by melting points of metals that must be melted in such a way as to avoid oxidization. This requires a neutral atmosphere, or that the sample be covered with some flux that will protect it. In the region above 1100° C. (*i.e.*, 1373° K.) there are two difficulties. The first is the necessity of reaching the high temperature in such a way that it can be controlled and measured and in addition there is the second difficulty of getting and maintaining the black-body conditions. This makes it very difficult for the man in the works to make a check on his pyrometer by a direct comparison with the black body at the temperature of the standard fixed points. It is necessary, therefore, to have a secondary method that is convenient and accurate enough for practical purposes. Such a method is described below.

MAINTAINING THE HIGH-TEMPERATURE SCALE.

As pointed out above, the high-temperature scale above 1300° K. is based upon the brightness of a black body at the temperature of the melting point of some one or more selected metals. Owing to their convenience, both as to their freedom from oxidization and as to the temperature of their melting points, gold and palladium are generally chosen. Both of these metals can be successfully melted in a platinum-wound black-body furnace. This is a very great convenience because the whole operation can be carried out in the open air, thus requiring no special precautions in maintaining either a vacuum furnace or a neutral atmosphere within the furnace. The black body at a particular temperature is the standard, whether the scale is determined by means of a total radiation or by means of an optical pyrometer. Most experimenters determine their temperatures in this region by means of an optical pyrometer.

For calibration at the palladium point the thermocouple of platinum, platinum-rhodium is not very reliable in that it will not remain constant, as it is used inside the platinum-wound furnace. This necessitates that each time a calibration is made the point be found by a melting-point determination. Any one who has used a black body for such purposes will know that this requires a great amount of work. Thus, while in the standardizing laboratory the scale should be maintained and checked by

actual measurements from the black body at the standard melting point, some other method is necessary in practice. Much time can be saved if a tungsten lamp with a filament of suitable size is standardized so as to have the same brightness for a particular part of the filament, when observed with the optical pyrometer, as the standard black-body furnace for one or more definite temperatures. Such lamps with single-loop wire filaments have been made, calibrated and used in this laboratory for this purpose for several years. The bend in the wire is made somewhat sharp and the centre of this loop used as a point of reference. This method has recently been employed by a number of research laboratories in this country using lamps calibrated at several currents for temperatures between 1200° K. and 2000° K.

For the highest accuracy the tungsten lamp that is to be used for calibration purposes should be standardized with an optical pyrometer, using a monochromatic screen (usually red glass), which is the same as that on the pyrometer to be compared, or corrections should be made for the difference. If the effective wave-lengths are known, this correction can easily be made.[5] For practical purposes, however, if similar red glasses are used the error will be quite small.

Tungsten filaments have been found to depart very markedly from Lambert's cosine law in their radiation.[6] To avoid error due to this cause, care must always be taken in determining the temperature of circular filaments to measure the brightness of the central part of the filament. For this reason, the pyrometer filament should always be parallel to the background filament. This, of course, requires that the pyrometer filament be much smaller than the image of the background filament used as a standard.

A ribbon tungsten filament with some mark, or a pointer to indicate the exact spot sighted upon, is sometimes used as a standard background against which to calibrate a pyrometer filament. The ribbon filament does not require as much magnification as the wire filament. A commercial form of the disappearing-filament pyrometer which has a small magnification can be calibrated, using such a ribbon filament, providing that either the filament is about 3 mm. wide or an extra lens is used between

[5] *Gen. Elec. Rev.*, **20**, p. 749, 1917.

[6] Worthing, *Astrophys. Jour.*, **36**, p. 345, 1912.

the filament and the pyrometer, so as to give a large image of the filament, this extra lens being of such size that it will not limit the cone of rays that enters the objective lens of the pyrometer. If a lens is used corrections must be made for its transmission, or the background lamp must be calibrated by means of the standard pyrometer with the lens before the lamp. The transmission of a lens is measured in much the same manner as is the transmission of a piece of absorbing or other glass. The method generally employed is to form an image with the lens whose transmission is being measured and then, taking care to see that the lens is not limiting the beam that enters the objective lens, to measure the brightness of this image and compare this brightness with that of the standard source as observed directly.

A ribbon-filament lamp and lens combination may be used to calibrate a Wanner-type of optical pyrometer. In this case, the image of the filament is to be focused on the slit of the pyrometer and readings made by obtaining the relation between the scale of the pyrometer and the temperature of the ribbon filament corrected of course for the transmission of the lens.

Several industrial plants have recently had such ribbon-filament lamps made and have either used them without or with an extra lens. Such a lamp or lamp and lens combination can be calibrated by a standard laboratory or they can be calibrated with the calibrated optical pyrometer before it has been used, and in this way any change in the calibration of the optical pyrometer can be detected.

With such lamps properly aged and properly calibrated, any one may maintain his own temperature scale for years, if the calibration does not extend higher than that of the palladium point and the standard lamp is not accidentally heated to a higher temperature.

An intercomparison of the temperature scales of the Bureau of Standards, the Research Laboratory at Schenectady, the Physical Laboratory of the University of Wisconsin, and Nela Research Laboratory was made by means of measurements made on a number of tungsten lamps. The temperatures of several tungsten lamps were very carefully measured in this Laboratory for different currents and sent to three other laboratories where the temperatures were measured for the same currents. The lamps were then returned to this Laboratory for a second check

on the temperatures. This gave an intercomparison of the temperature scales that are in use in the different laboratories.

In each of the laboratories the temperatures were measured by means of a disappearing-filament optical pyrometer, using red glass as the monochromatic screen. As the different laboratories used red glasses having slightly different effective wave-lengths, a small correction was necessary to reduce the temperatures to the same wave-length. Such a correction would have been unnecessary if the sources whose temperatures were measured had been black bodies. A summary of the results, using the data on but three of the six lamps, is given in Table III. The data obtained on the other three lamps are about as those presented in the table, as can be seen by a reference to the original paper.

TABLE III.

Results of Intercomparison of Temperature Scales.

$c_2 = 14350\mu$ deg. $\lambda = 0.665\mu$ Melting point of Au = 1336° K. (Pd = 1828° K.)

Lamp	Nela Research Laboratory (10-17-16)	Research Laboratory of General Electric Company (Schenectady)	Nela Research Laboratory (11-28-16)	Bureau of Standards	Nela Research Laboratory (4-2-17)
T—16B	.....	....	1429	1431	1427
T—30C	1813° K.	1813	1813	1814	1813
T—30C	.....	2307	2304	2302	2303
T—30C	2756	2752	2752	2762	2752

	Nela Research Laboratory (3-27-16)	Physical Laboratory University of Wisconsin		Nela Research Laboratory (7-14-17)
		C. E. M.	G. R. G.	
T—17—C	1810	1813	1816	1810
	2193	2197	2202	2196
	2490	2506	2516	2497

Of the lamps used in the intercomparison all, except one, had flat filaments about 3 cm. long, and about 1½ mm. wide. The exact point at which it was desired to have the temperature measured was indicated either by a pointer, a notch in the supporting lead, or a small notch in the filament itself. Three of the lamps were gas-filled, and the other two were of the vacuum type. The other lamp, a gas-filled lamp, had a 20-mil. (0.5 mm.) filament in the shape of a hairpin loop. As the loop was rather sharp, the exact point at which it was desired to have the temperature measured was easily indicated. This Laboratory is planning to

send similar lamps to several other laboratories for comparisons in the near future.

An inspection of Table III will show that the results by the Bureau of Standards, Research Laboratory of the General Electric Co., Schenectady, and Nela Research Laboratory, agree very well for the entire range studied. The results obtained in the Physical Laboratory of the University of Wisconsin do not agree quite so well, but even here the differences are quite small.

The above results show two things: first, that the temperature scales in use in the four laboratories that made the intercomparison are in very good agreement, and second, that the temperature can be satisfactorily measured from such lamps.

Six years ago a tungsten lamp with a single hairpin loop (ten-mil. filament) was standardized in this Laboratory so as to have the same brightness as observed with the pyrometer as the black body at the temperature of melting palladium, and has been used ever since as one of two standards of temperature. This lamp has also been used as a brightness standard against which to calibrate pyrometer filaments for brightness. It has not been operated at a higher temperature than that corresponding to the brightness of a black body at the palladium point. When it was first calibrated, six years ago, the current required for this brightness was 4.087 amperes. A recent recalibration of this lamp indicated that the current should be 4.095. The other standard has been in use for about four years and the recent calibration showed no change in its current for this brightness. The change in current noted in the first lamp corresponds to a little over one degree centigrade. Thus, if the temperature scale had been maintained for the six years by this one lamp alone, the error would amount to about 1° C. This for most work would be negligible.

PRECAUTIONS IN USING OPTICAL PYROMETERS.

The Disappearing-filament Type.—In working with an optical pyrometer of the disappearing-filament type, experience has shown that several precautions are necessary in order to avoid errors. If an optical pyrometer of this type is constructed without a limiting diaphragm between the objective lens and the pyrometer lamp, an error will be made if the position of the objective lens is changed, even though there is a limiting diaphragm between the

pyrometer lamp and the eyepiece. In other words, the current required through the pyrometer filament for an apparent brightness match with a particular source is a function of the angle that the cone of rays from the objective lens makes at the pyrometer filament. This has been found to be due to light from the source being diffracted [7] around the pyrometer filament.

It has often been assumed by different experimenters that the pyrometer filament and the background source were at the same brightness (excepting for small differences due to lens absorption, etc.), when there was an apparent brightness match between them. This is not the case since part of the light that appears to come from the pyrometer filament comes from the source being investigated by being diffracted by the filament. These errors can be avoided by keeping the cone of rays from the objective lens to the pyrometer filament as well as the cone of rays entering the eyepiece, fixed for any particular set of measurements. This can readily be done by having fixed limiting diaphragms between both the objective lens and the pyrometer filament, and between the pyrometer filament and the eyepiece, the latter diaphragm being the smaller. It has been shown [8] that conditions can be so arranged that the pyrometer filament will be as much as 1.6 times as bright as the source being investigated, and still as observed through the eyepiece there will be an apparent brightness match. On the other hand, by increasing the size of the cone of rays that reaches the pyrometer filament to a sufficiently large value it will be found that there will be an apparent brightness match when the pyrometer filament is only about 0.9 as bright as the source. The former condition would probably not be met with in practice since the cone of rays is rather small, being about such as would be subtended by an opening 0.2 mm. in diameter at a distance of 50 cm. In the latter case the cone is about such as would be subtended by an opening 1 cm. in diameter at a distance of 50 cm.

To get a perfect brightness match the different parts of the pyrometer should be so lined up that there is approximately axial symmetry, and the resolving power of the eyepiece used should not be so great as to prevent the practical disappearance of the pyrometer filament against the background image. This

[7] *Phys. Rev.*, N. S., 4 p. 163, 1914.
[8] *Loc. cit.*

image should be large in comparison with the pyrometer filament for reasons given below.

If a rotating sector is used to cut down the apparent intensity of the background, care must be taken as to the location [9] of the sector. There is a very marked difference in the results of temperature measurements, depending upon whether the sector is located near the objective lens or as near as possible to the pyrometer lamp. There is also a difference depending upon the relative positions of the openings in the sector and the source, providing the source is a lamp filament. If a sector of small transmission is mounted near the lens and so placed that the openings of the sector are parallel to the axis of the background filament when the sector is passing across the centre of the lens, the definition will be very bad, while if the openings of the sector are turned through 90° so that they are perpendicular to the axis of the filament, the definition will be quite good, but not so good as if the sector is located near the pyrometer lamp. When the rotating sector is located near the pyrometer lamp, the definition is good and practically independent of the relative position of the opening of the sector. If a very large source is used no such effect is noted. Using a pyrometer calibrated against such a large background and thus independent of the position of the sector to measure the brightness temperature of a small tungsten filament, large variations in temperature were found when different sectors were used near the objective lens. No such differences were found when the sector was located near the pyrometer filament.

In Table IV are given results of a test showing the effect of the position of the sector. A fifteen-mil (0.381 mm.) tungsten lamp, operated at a brightness temperature of about 2275° K., was used as a background, and readings were made on the current through a 2.5-mil tungsten pyrometer filament, for an apparent brightness match with a sector having two one-degree openings. From the table it can be seen that the position of a sector of this size can cause an error of about 14° K. for this condition if care is not taken as to its location.

If in place of a rotating sector an absorbing screen is used to cut down the apparent intensity, care must be used, due to the fact that absorbing screens are not neutral-tint, and thus their transmission depends to some extent upon the temperature of the source being investigated. This change in transmission may

[9] Hyde, Cady and Forsythe, *Astrophys. Jour.* **42**, p. 303, 1915.

TABLE IV.

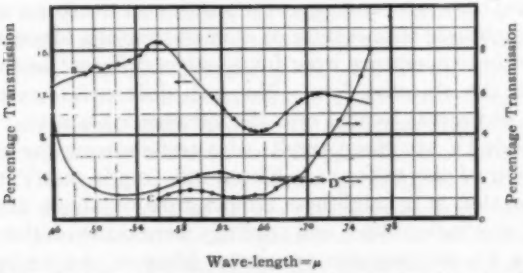
Errors in Temperature Measurements Due to Improper Location of Sector.

	Position of a 2-degree sector			
	Near lens		Near pyrometer lamp	
	Opening of sector parallel to background filament	Opening of sector perpendicular to background filament	Opening of sector parallel to background filament	Opening of sector perpendicular to background filament
Current in amperes through pyrometer filament for brightness match.....	0.3332	0.3354	0.3357	0.3357
Apparent relative brightness.....	.939	.995	1.000	1.000
Temperature of background for these readings.....	2263° K.	2275° K.	2277° K.	2277° K.

be enough to cause a considerable error. The transmission of an absorbing screen also depends upon its own temperature.

Another point that should be noted [10] is that when two absorbing glasses are used together, the transmission of the two used together is not, in general, the product of their separate trans-

FIG. 1.



Spectral transmission of various absorbing glasses. Curve *B*, Jena absorbing glass 1.5 mm. thick. Curve *C*, Noviweld obtained from Corning Glass Works; shade about 6. Curve *D*, Leeds & Northrup absorbing glass made of purple and green glass.

missions. This is due to the fact that the spectral transmission of absorbing screens is not in general independent of the wave-length.

In Fig. 1 are shown the spectral transmissions of a piece of noviweld and of a piece of a Jena absorbing glass. Either one

[10] Foote, *Bur. Stds. Bull.*, Vol. 12, p. 500, 1916.

of these glasses is nearly enough neutral tint for use with the red glasses whose transmission curves are shown by curves B, C, and D in Fig. 3. The noviweld absorbing glass and all the samples of Corning high-transmission red glasses were obtained from Mr. H. P. Gage, of the Corning Glass Works, Corning, New York. This absorbing glass is made in different shades with transmissions, when used in connection with red glass, ranging from less than one per cent. to several per cent.

If a red glass is used in the eyepiece, the ratio of the brightness of the source as observed through both the red glass and the black glass, to the brightness of the same source as observed through the red glass alone, is called the total transmission for the particular temperature. Without a red glass, using the entire visible spectrum, it is generally very hard to make such measurements owing to the color differences introduced by even the best absorbing glasses, but with a good red glass in the eyepiece, such transmission measurements can be made easily.

The total transmission of the absorbing glass when used with a red glass can be calculated for any black body distribution by the following formula taken from Preston's Theory of Light:

$$T_B = \frac{\int_0^\infty J(\lambda T) V_\lambda t'_R t'_B d\lambda}{\int_0^\infty J(\lambda T) V_\lambda t'_R d\lambda} \tag{6}$$

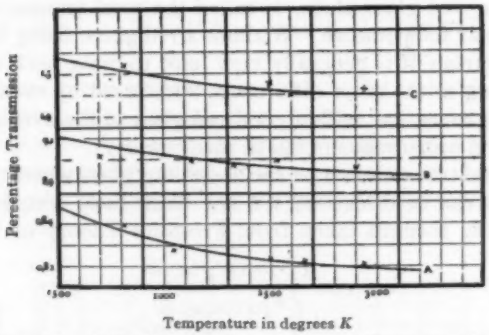
where $J_\lambda \, d\lambda$ = black-body energy for interval λ to $\lambda + d\lambda$. V_λ = visibility, t'_R and t'_B = spectral transmission of the red and absorbing glasses, respectively. It is very evident that if the spectral transmission of the absorbing glass is different for different wavelengths, the total transmission will be a function of the temperature of the source under investigation.

In Fig. 2 is shown as a function of the temperature of the source, the total transmission for red light of the absorbing glasses whose spectral transmission is given by curves B and C, Fig. 1. The measured points were determined by the author, and the calculated values were obtained by means of equation (6) by making use of an average visibility curve [11] for this spectral region. Transmission values were also calculated, using the writer's visibility curve. Values thus obtained, using the two different visibility curves, differ from each other by only a small fraction of a per cent.

[11] *Astrophys. Jour.*, **48**, p. 87, 1918.

The Effective Wave-length of the Monochromatic Screen.—An optical pyrometer can be calibrated and so used as to make unnecessary a knowledge of the extent to which the screen is monochromatic. To do this requires a black-body furnace that can be operated at various temperatures up to the highest temperature for which the pyrometer is to be used. However, if any attempt is made to use Wien's equation to extend the temperature scale, either above or below that of the standard furnace by the use of rotating sector discs or absorbing glasses, that is, to

FIG. 2.



Temperature in degrees K

Total transmission of absorbing glasses, as a function of temperature when used with red glass No. 4512—5.8 mm. thick. Curve *A*, two pieces Jena absorbing glass. *B*, one piece Jena absorbing glass. *C*, Noviweld glass from Corning Glass Works. Curves drawn through points calculated from equation $T_B = \frac{\int_0^\infty V_\lambda T_\lambda T'_R T_B \, d\lambda}{\int_0^\infty V_\lambda T_R \, d\lambda}$ Crosses represent values of transmission obtained with optical pyrometer.

find the temperature of a black body whose brightness, as seen through the pyrometer, is, say, ten times (assuming a sector or absorbing-glass transmission of one-tenth), that of a black body whose temperature can be measured directly, a knowledge of what wave-length to use is necessary. A knowledge of the wave-length that must be used may also be necessary if the pyrometer is used to measure the temperature of non-black bodies. The wave-length which must be used is called the "effective wave-length." [12] In using the pyrometer, it is the integral luminosities through the red glass that are compared, and for this reason the effective

[12] *Astrophys. Jour.*, **42**, p. 294, 1915.

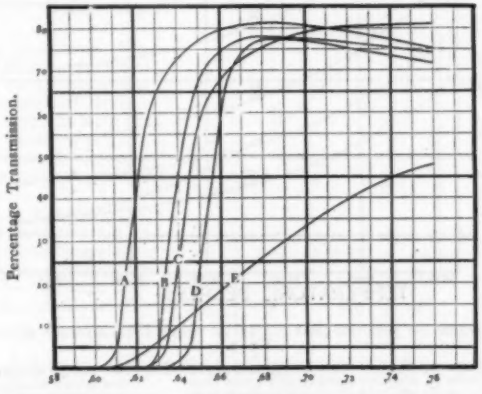
wave-length of the red glass screen for a certain temperature interval has been defined as the wave-length such that, for the definite temperature interval for a black body, the ratio of the radiation intensities for this wave-length shall equal the ratio of the integral luminosities through the screen used.

Knowing the spectral transmission of the red glass, it is possible to calculate the effective wave-length λ_e for any temperature interval by means of the following equation:

$$\left[\frac{J(\lambda T_1)}{J(\lambda T_2)}\right]_{\lambda_e} = \frac{\int_0^\infty J(\lambda T_1) V_\lambda t'_R d\lambda}{\int_0^\infty J(\lambda T_2) V_\lambda t'_R d\lambda} \tag{7}$$

These integrals can be computed by the step-by-step method with

FIG. 3.

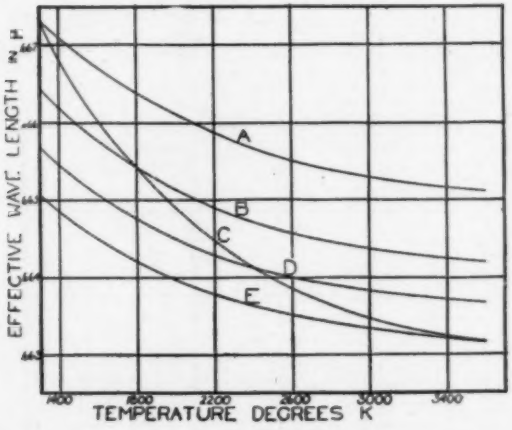


Spectral transmission of various red glasses. Curve *C* for Jena red No. 4512, 2.93 mm. thick. Curve *E* for Jena red No. 2745, 3.2 mm. thick. Curve *A* for Corning high transmission red, marked 150 per cent., 5 mm. thick. Curve *B* for Corning high transmission red, marked 50 per cent., 5 mm. thick. Curve *D* for Corning high transmission red, marked 28 per cent., 6 mm. thick.

sufficient accuracy for this purpose. Using equation (7) the effective wave-length was calculated for the red glass whose spectral transmission is shown by curve C, Fig. 3, for a number of temperature intervals and plotted as shown in Fig. 4. By connecting the points where the curve for the effective wave-length from any particular temperature crosses the same temperature ordinate, a curve is obtained (curve C, Fig. 4) that

gives the limiting effective wave-length for any particular temperature. To show how these curves may be used, the effective wave-length for a couple of temperature intervals will be found. The effective wave-length between 1800° K. and 2900° K. is given by the ordinate of the point where the 1800° K. curve crosses the 2900° K. ordinate; that is, it is 0.6644μ. For the range between 2100° K. and 2900° K. the effective wave-length is likewise given by the point where the 2100° K. curve would

FIG. 4.



Effective wave-length for Jena red glass No. 4512. Spectral transmission shown in curve *C*, Fig. 3.

Curve *A*, effective wave-length from 1300 °K to other temperatures.
Curve *B*, effective wave-length from 1800 to other temperatures.
Curve *D*, effective wave-length from 2400 to other temperatures.
Curve *E*, effective wave-length from 3000 to other temperatures.
Curve *C*, limiting effective wave-lengths.

cross the 2900° K. ordinate. However, in this case the 2100° K. curve is not drawn, but will have to be imagined as drawn parallel to the 1800° K., one point of its position being determined by where the curve crosses the 2100° K. ordinate. The effective wave-length for this interval is 0.6641μ. It can be seen from the figure that the effective wave-length for any temperature interval is given quite closely by the mean of the limiting effective wave-length for the two temperatures.

In Table V are given the errors that would result due to

TABLE V.

Changes in Temperature of 2400°, and 3000° K. extrapolated from 1800° K. as Initial Temperature, Using Wien's Equation, Due to Various Changes.

$c_2 = 14350\mu$ deg.
$\lambda = 0.665\mu$

Variation leading to error	Percentage change			Actual change		
	1800° K.	2400° K.	3000° K.	1800° K.	2400° K.	3000° K.
One per cent. change in initial temperature.....	1%	1.3%	1.7%	18°	32°	50° K.
3° K. change in initial temperature.....	..	....	....	3°	5°	8°
Using a λ that is one per cent. in error.....	..	0.3	0.7	....	8°	20°
0.001μ error in λ.....	..	0.05	0.10	....	1.2°	3°
If in extrapolating the λ_e of red glass between 1300° and 1800° K. is used. See Fig. 4.....	..	0.1	0.3	....	2.4°	7.5°
Calibrating pyrometer filament against tungsten lamp as background that was standardized with a red glass different from the one used in the pyrometer being calibrated. Suppose λ_e to change from 0.665μ to 0.650μ.....	..	....	....	3.5°	....	....
One per cent. error in value used for transmission of sector or absorbing glass...	..	0.11	0.14	....	2.8°	4.2°
One per cent. variation in the current through 2½-mil. pyrometer filament...	0.5	0.7	0.8	9°	16°	25°

various changes. From this table it can be seen what error in temperature would result in an error in the effective wave-length.

The Wanner Pyrometer.—In the Wanner optical pyrometer the beams of light from the comparison source and the source studied are so arranged that, by the use of a polarizing device, the two beams, as viewed through the eyepiece, are polarized in a plane at right angles to each other. By rotating another nicol located in the eyepiece, the two sources can be made to appear the same in brightness. When this form of pyrometer is used there is always the danger of error due to polarized light from the source. If readings are taken with the pyrometer in two planes at right angles to each other, the chance for error due to this cause will be greatly lessened. There must be a very large source

in order to fill the entering cone of the pyrometer and the instrument must be pointed directly at the source being studied. It is sometimes difficult to be sure that these two conditions are fulfilled. For good working conditions, the different parts of the instrument must be very carefully lined up. If this is not the case there will be either an overlapping of the two fields, or there will be a dark line between them.

All of the possibilities of errors due to absorbing screens discussed under the disappearing-filament type, also apply to this pyrometer.

If a red glass is used to obtain monochromatic light, the same care concerning the red glass is necessary, as was pointed out above. If the monochromatic radiation is obtained by the use of a direct-vision prism, the slits must not be too wide (as expressed in wave-length interval transmitted), or the same, or even greater error, will result. The comparison lamp must be kept in position and stray light kept from the different paths.

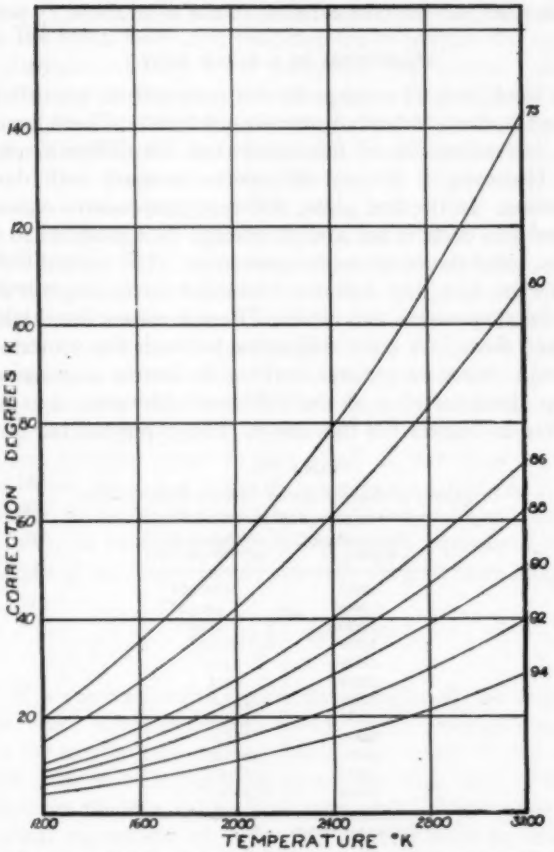
The Le Chatelier Pyrometer.—In this form of optical pyrometer the comparison source is off at one side and the comparison is made between this and the source being studied by means of a photometer cube or a mirror. This pyrometer is made up in several forms. They all, in general, require a rather large source for observation, and it is also sometimes difficult to be sure that the instrument is pointed directly at the source.

The same care is required concerning the absorbing screen and the monochromatic screen, as was pointed out under the heading of the disappearing-filament pyrometer.

Corrections for the Transmission of Window.—When measuring temperatures with an optical pyrometer it is often necessary to make observations through a glass or other window, or through the lamp bulb. This, of course, makes necessary a correction to the temperature, which correction depends upon the transmission of the glass or other window and upon the temperature measured. This correction can be calculated by the use of Wien's equation and the transmission of the window. In Fig. 5 are shown the corrections for various transmissions. From these curves the correction for any transmission from about 100 per cent. to 75 per cent. and for a temperature range from 1200° K. to 3200° K. can readily be found. These corrections apply to wave-length 0.665μ. If a different effective wave-length is used, corrections

can easily be made. From Wien's equation it can be shown that the percentage variation in this correction is equal to the per-

FIG. 5.



Corrections to be added for various transmissions of windows for different observed temperatures. Each curve shows the corrections for the per cent. transmission marked at the end of the curve in the right margin.

centage variation in the wave-length used, *i.e.*, if a wave-length one per cent. greater than 0.665μ is used the correction will be one per cent. greater than the one given in the curves.

The transmission of a clean lamp bulb for red radiation has been found to be 90 per cent. or slightly more. If the glass is blackened, it may be lowered to 88 per cent. or even lower, depending upon the amount of the blackening. A mica window was measured and the transmission found to be about 75 per cent.

BRIGHTNESS OF A BLACK BODY.

The black body at some particular temperature has often been suggested as the standard of intensity of light. There have been several determinations of this brightness for different temperatures. However, it is very difficult to compare such data for two reasons. In the first place, different temperature scales have been used and there is not always enough data included to determine just what the temperature scale was. The second difficulty is due to the fact that different observers have, in general, different visibility curves and obtain different values for the brightness when there is a color difference between the sources being compared. Since in general nothing is known concerning the visibility characteristics of the different observers, it is almost impossible to correct for this error. The experimental values [13]

TABLE VI.

Brightness of Black Body at Various Temperatures.

Temperature degrees K.	Brightness in candles per cm.²
1000	0.00251*
1200	.0132*
1400	.231*
1600	2.03*
1800	11.3
2000	44.3
2200	137.6
2400	354.0
2600	785.0
3000	2892.0*
3600	11880.0*
5000	87300.0*

* Calculated values.

of brightness given in Table VI are on the temperature scale given above, and in order to avoid possible errors arising from individual idiosyncrasies of vision, the candlepower scale on which the values given in the table were based was that given for

[13] Hyde, Forsythe and Cady, *Phys. Rev.* II, 13, p. 45, 1919.

a tungsten lamp by Middlekauff and Skogland.[14] The procedure of calibration consisted in transferring the photometer and comparison lamp to a standard photometer bench, and substituting a tungsten vacuum lamp for the black body. The comparison lamp was then brought successively to the various voltages used against the black body and readings made against the tungsten lamp. In cases of large color difference blue screens calibrated at the Bureau of Standards were used. The relative candlepowers of the standard tungsten lamp were computed from the data of Middlekauff and Skogland, and hence the candlepowers of the comparison lamp, and so the brightnesses of the black body were obtained on this same scale. The candlepower scale of the Bureau of Standards was established originally by eight observers. Very fortunately the average visibility of these eight is not markedly different from that of the average of 125 observers as determined by Coblentz. At any rate these values are given on a scale of candlepower that is known and does not depend upon observation of a single observer whose visibility curve is unknown. In the table the values of the brightness are given in candles per square centimeter at various temperatures. The values between 1700° K. and 2600° K. are observed values, and the values above and below these limits are calculated from the distribution of energy in a black body as given by Wien's equation, the visibility of radiation, and the mechanical equivalent of light. This required an evaluation of the following definite integral:

$$(8) \qquad B_0 = \frac{1}{\pi M} \int_0^\infty J(\lambda T) V_\lambda d\lambda$$

where M is the mechanical equivalent of light, B_0 the brightness, and the other terms as defined above. This integral was evaluated by the step-by-step method, using steps where $d\lambda$ was 0.005μ. In some other work it has been shown that with steps of this size the error in relative values was negligible. The value of the mechanical equivalent of light, M, used is 0.00150 watts per lumen. The values of the other constants are given in the summary below. The mechanical equivalent of light used here is the one adopted by the Illuminating Engineering Society. It is the value found from the experimental value of the brightness given

[14] *Bull. Bur. of Stds.*, 11, p. 483, 1915.

TABLE VII.

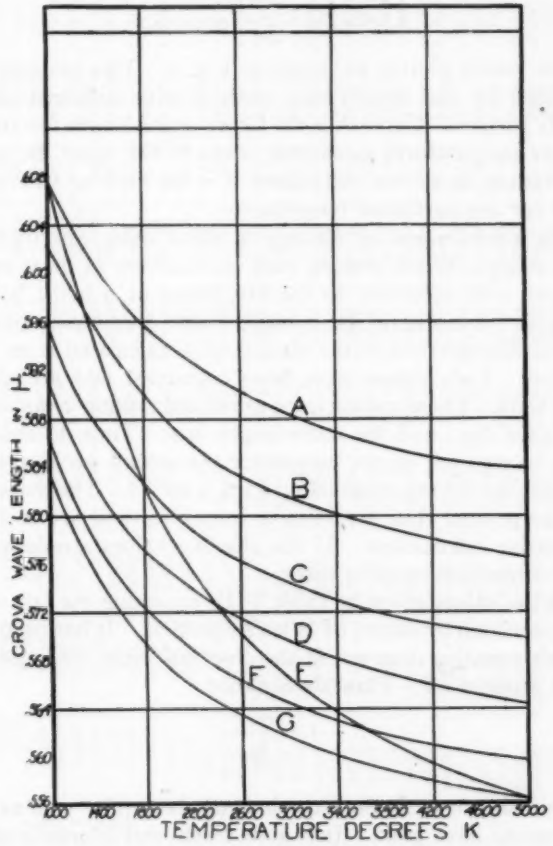
Investigators	Recommended values of the mechanical equivalent of light in watts per lumen	Visibility data	Method
P. G. Nutting...	0.00120 ± 0.00005	Nutting	Measurements of total radiant flux and total luminous flux from acetylene. Coblentz energy curve for acetylene used.
H. E. Ives.....	0.00160 ± 0.00003	.....	(1) Measurement of total radiant flux and total luminous flux from a monochromatic source.
		Ives – Kingsbury curve from equation Ives – Kingsbury	(2) Measurement of radiant flux through luminosity filter and integral luminous flux direct from the source.
	Secondary value 0.00154	.....	(3) Measurement of brightness of a platinum-wedge black body at the melting point of platinum taken as 2037° K.
W. W. Coblentz	0.00162 ± 0.00005	Coblentz and Emerson	Several methods using observations by himself and others. (1) Computation using preliminary data on brightness of a black body by Hyde, Forsythe and Cady. (2) Direct measurement on monochromatic radiation by Coblentz, Ives and Kingsbury. (3) Measurements on incandescent lamps by Coblentz and Emerson.
Hyde, Forsythe and Cady	0.00150 ± 0.00005	Hyde, Forsythe and Cady	Direct measurement of brightness of carbon and platinum tube black bodies.

in Table VI. In Table VII are given the recent determinations of the mechanical equivalent of light, together with a statement of the method used.

Crova Wave-length.—Crova and others have shown that the relative brightness of a black body at different temperatures varied

as the brightness of some particular wave-length interval near the centre of the visible spectrum. This wave-length is not constant as was first thought, but varies in such a way that the

FIG. 6.



Crova wave-length.

Curve *A*, Crova wave-length from 1000 to other temperatures.
Curve *B*, Crova wave-length from 1300 to other temperatures.
Curve *C*, Crova wave-length from 1800 to other temperatures.
Curve *D*, Crova wave-length from 2400 to other temperatures.
Curve *F*, Crova wave-length from 3600 to other temperatures.
Curve *G*, Crova wave-length from 5000 to other temperatures.
Curve *E*, limiting Crova wave-length.

Crova wave-length moves towards a shorter wave-length for higher temperatures.

The Crova wave-length was calculated by means of the following formula:

$$\left[\frac{J(\lambda T_1)}{J(\lambda T_2)}\right]_{\lambda_c} = \frac{\int_0^\infty J(\lambda T_1) V_\lambda d_\lambda}{\int_0^\infty J(\lambda T_2) V_\lambda d_\lambda} \tag{9}$$

and the results plotted as shown in Fig. 6. This integral can be calculated by the step-by-step method with sufficient accuracy for this purpose. Curve A is the Crova wave-length for 1000° K. to other temperatures as shown; curve B for 1300° K. to other temperatures as shown, etc.; curve E is the limiting Crova wave-length for any particular temperature.

The Distribution of Energy in the Visible Spectrum of a Black Body.—When making such calculations as those referred to above with reference to the brightness of a black body and the Crova wave-length, use is made of the distribution of energy in the visible spectrum of the black body as calculated from Wien's equation. Such values have been calculated and are given in Table VIII. These values were calculated, taking c_1 as unity, c_2 as 14,350μ deg., and the wave-length in μ. Thus, to reduce the values to ergs per square centimeter per second per μ, they must be multiplied by the value of c_1 ($3.7_2 \times 10^{-12}$). The values [15] as they are printed thus are relative intensities and are to be used for relative calculations. If the absolute values are desired, the above corrections must be made.

All the values given in Table VIII, excepting the last column, were calculated by means of Wien's equation. It has been shown that this equation does not fit the observed facts for large values of the product λT. Planck's equation

$$J_\lambda = \frac{c_1 \lambda^{-5}}{e^{\frac{c_2}{\lambda T}} - 1} \tag{10}$$

has been shown to fit the experimental data quite well as far as experiments have gone. It is quite likely that Planck's equation would better represent the experimental values at very high

[15] If above values for the spectral distribution are reduced to actual energies care must be taken to introduce the value for $d\lambda$. Wien's equation, as generally used, gives the value of the intensity rather than the actual energy. To reduce to energy the value of $d\lambda$ must be introduced.

TABLE VIII.

Relative Energy of a Black Body at Various Temperatures and Various Wave-lengths.

λ	1000 $\times 10^{-13}$	1200 $\times 10^{-12}$	1400 $\times 10^{-12}$	1600 $\times 10^{-11}$	1800 $\times 10^{-10}$	2000 $\times 10^{-9}$	2200 $\times 10^{-8}$	2400 $\times 10^{-8}$	2600 $\times 10^{-7}$	3000 $\times 10^{-7}$	3600 $\times 10^{-6}$	5000 $\times 10^{-5}$	5000 $\times 10^{-5}$
0.40	26	101	727	1787	2158	1584	809	3148	994	6256	4612	7465	7482
.41	54	186	1200	2728	3101	2168	1064	4005	1230	7403	5056	7872	7878
.42	111	330	1926	4073	4368	2915	1378	5026	1502	8663	5782	8244	8252
.43	214	568	3024	5949	6037	3856	1757	6220	1813	10038	6407	8591	8601
.44	416	954	4642	8519	8204	5022	2212	7608	2163	11520	7051	8911	8923
.45	766	1559	6946	11976	10965	6449	2747	9194	2555	13111	7724	9208	9222
.46	1375	2490	10210	1655	1444	8171	3373	10995	2989	14798	8373	9475	9492
.47	2396	3388	14740	2251	1875	10226	4096	13020	3464	16580	9041	9716	9737
.48	4091	5947	2089	3014	2403	12650	4923	1528	3982	18450	9712	9933	9957
.49	6776	8933	2913	3981	3042	1548	5860	1777	4544	19975	10378	10123	10151
.50	1099	13169	4003	5189	3808	1876	6914	2050	5143	22410	11039	10286	10319
.51	1746	1902	5417	6682	4714	2251	8087	2348	5784	24480	11688	10429	10466
.52	2723	2709	7234	8502	5778	2677	9382	2669	6464	26614	12329	10544	10586
.53	4174	3800	9545	10706	7016	3158	10810	3015	7180	28780	12954	10639	10686
.54	6267	5255	12430	13340	8441	3695	12370	3383	7930	30980	13563	10712	10764
.55	9273	7171	16020	16460	10072	4292	14050	3775	8712	33205	14148	10762	10820
.56	13490	9666	2033	20120	11925	4952	1587	4190	9524	35444	14715	10797	10862
.57	1935	12870	2577	2439	14006	5674	1782	4624	10364	37675	15260	10813	10882
.58	2741	16940	32214	2934	16347	6464	1990	5079	11234	39920	15782	10811	10887
.59	3826	2205	3990	3499	18943	7318	2211	5553	12112	42058	16280	10793	10877
.60	5275	2842	4901	4146	21820	8240	2443	6046	13012	44356	16752	10763	10853
.61	7184	3627	5967	4876	2498	9230	2688	6553	13936	46542	17197	10716	10815
.62	9684	4588	7215	5699	2843	10285	2945	7077	14868	48702	17618	10660	10767
.63	12915	5753	8660	6618	3218	11410	3212	7612	15808	50804	18010	10590	10702
.64	17041	7153	10321	7640	3625	12600	3491	8162	16742	5288	18332	10510	10632
.65	22261	8824	12213	8772	4064	13850	3779	8720	17691	5490	18713	10420	10547
.66	28820	10802	14375	10016	4533	15170	4076	9285	18642	5699	19024	10323	10457
.67	36979	13129	16814	11382	5036	16550	4382	9861	19590	5877	19313	10218	10362
.68	47060	15859	19555	12870	5571	1799	4696	10441	2054	6064	19573	10101	10252
.69	59398	19008	22617	14480	6138	1949	5016	11029	2148	6240	19809	9985	10141
.70	74405	22667	26019	16230	6736	2104	5342	11614	2240	6409	20022	9861	10024
.71	92512	26860	29792	18100	7367	2265	5674	12198	2321	6574	20207	9732	9903
.72	11419	31650	33951	20120	8030	2430	6012	12786	2422	6732	20368	9600	9776
.73	14005	37068	38513	22270	8720	2600	6352	13372	2512	6882	20511	9462	9652
.74	17063	43217	43486	24560	9443	2773	6696	13957	2598	7023	20627	9323	9514
.75	20663	50130	48910	27000	10195	2951	7042	14535	2684	7160	20725	9179	9377
.76	24874	57856	54809	29570	10971	3133	7391	15110	2767	7287	20828	9035	9239

All of the values in this table were calculated by means of Wien's equation with the exception of the last column, headed "5000 $\times 10^{-5}$" and that is calculated by means of Planck's equation.

temperature if such observations could be made. In Table IX are given the correction factors to be used to convert the values obtained by calculation, using Wien's equation to values that would be given by Planck's equation for different values of the product λT.

TABLE IX.

Ratio Between Planck's and Wien's Equations for Different Values of λT.

λT	Er./Ew
2000	1.0008
2500	1.003
3000	1.008
4000	1.028
5000	1.056

In the last column of Table VIII are given the values for the relative intensity for 5000° K. corrected to Planck's equation by means of these factors. Using Planck's instead of Wien's equation in calculating the brightness of the black body gives an increase in the calculated brightness of about 0.7 per cent. for a temperature of 5000° K., and an increase of less than 0.1 per cent. for the temperature of 3600° K. The error would of course be negligible for lower temperatures.

RADIATION CONSTANTS.

When referring to the different radiation laws mention was made of different constants. There has been a great deal of work done on these radiation constants. A very good summary of all this work is given by Coblentz.[16] Taking account of all the work and the accuracy of the different determinations, the best value for c_2 of the Wien-Planck equation is 14,350μ deg. Some later work seems to indicate that this value is slightly too high. However, until more data are available it does not seem necessary to make a change in the value. The value of the constant δ of the Stefan-Boltzmann total radiation laws which best fits all the experimental data is

$$5.7\times10^{-12} \text{ watts cm.}^{2} \text{ deg.}^{-4}$$

Using these values for c_2 and δ the value of c_1, the Wien-Planck equation is found to be

$$3.72\times10^{-12} \text{ watts cm.}^{2}$$

[16] *Bull. of Bur. of Stds.*, xiii, p. 459, 1916.

EMISSIVE POWERS OF VARIOUS RADIATORS.

All solid bodies when heated send out radiation and the intensity of this radiation has been found to depend upon the substance used and upon the temperature. It also depends upon the condition of the surface, that is, whether the surface is rough or smooth. The words rough and smooth are used in a relative sense, for a surface that is rough for one wave-length interval may well be rated as smooth for another. Since the intensity of the radiation in the various wave-lengths depends upon the temperature and upon the surface studied, the actual temperature can be calculated if the relation between the intensity of the radiation and temperature is known for this surface. As mentioned above, this relation between temperature and intensity of radiation for the black body is represented by Planck's equation. For values of the product λT below 3000 this equation may be reduced to Wien's form which is much easier to use in calculation. All other bodies that owe their brightness to temperature alone when at a particular temperature send out less radiation than does a black body at the same temperature. Thus, equation 4, if applied to other than a black body, would be written

$$J_\lambda = \varepsilon c_1 \lambda^{-5} e^{-\frac{c_2}{\lambda T}} \tag{11}$$

where ε (less than unity) is the spectral emissive power of the substance under consideration. ε will depend upon the wave-length, the conditions of the surface and upon the temperature. In Table X are given the emissive powers of some of the metals used as standards.

From the definition of a black body and Kirchhoff's law it follows that all other bodies that owe their brightness to thermal causes alone are less bright when at a particular temperature than a black body at the same temperature. Thus, if the temperature of any hot non-black body is calculated from a measurement of its brightness, as though it were a black body, values that are lower than the true temperature will be obtained. The temperature obtained, however, is the temperature that a black body must have in order to have the same brightness for the particular wave-length interval as the body being investigated. As the brightness thus measured corresponds to some particular wave-length interval the temperature obtained corresponds to that particular wave-length interval. The difference between the true

TABLE X.

Emissive Powers of Various Metals.

Metal	Temperature in °K.	Emissive power
Silver	1200	0.044
Gold	1300	0.14
Copper	1200	0.100
	1300	0.114
Platinum	1000	0.36
	1400	0.32
	1800	0.29
Palladium	1000	0.36
	1400	0.33
	1500	0.31
Iridium	2000	0.30
Nickel	1400	0.36
	1700	0.36
Iron	1300	0.379
	1500	0.375
	1800	0.360
Molybdenum [1]	1300	0.56
	2600	0.62
Tungsten [2]	1200	0.458
	1600	0.449
	2000	0.439
	2400	0 429
	2800	0.419
	3400	0.405
	3675	0.398

The values for the different emissive powers given in the above table are taken from a table by Burgess and Waltenberg (*Bull. Bur. of Stds.*, 11, p. 591, 1915) excepting as noted.

(1) Mendenhall and Forsythe, *Astrophys. Jour.*, 38, p. 380, 1913.

(2) Worthing, *Phys. Rev.*, N. S., X, p. 377, 1917.

temperature and the temperature thus obtained varies from a few degrees for such a substance as untreated carbon to more than 200° C. for such a metal as polished platinum at its melting point.

This temperature of non-black bodies has been called the black-body temperature, the apparent temperature and the brightness temperature. Arguments might be made in favor of any of these names for this temperature. It is the apparent temperature or the black-body temperature in that it is the temperature that is obtained directly from brightness measurements by assuming that the body behaves like a black body. However, the color temperature, as will be defined below, is also an apparent temperature, or a black-body temperature, although based upon entirely different principles. In this Laboratory, because it is the brightness that is measured, this temperature is called the brightness temperature and is referred to a particular wave-length. There is, to be sure, another temperature that might be called the brightness

temperature; that is, a temperature based upon the intensity or brightness of the total light radiated. However, both of these temperatures are based on the same principle. In using any one of these names, one must, if accuracy is desired, specify the wave-length to which the temperature belongs, as any non-black body has as many different brightness temperatures as there are wave-lengths to be considered. This brightness temperature, of course, varies for different wave-lengths. If a red glass is used as a monochromatic screen, the wave-length used varies with the temperature. It has been shown that the brightness temperature should be ascribed to the effective wave-length of the screen used for black-body radiation for the temperature interval from T equal to the brightness temperature to T_c, the color temperature of the source being investigated.

The color temperature of a particular source has been defined as the temperature of a black body, which has the same distribution of energy in the visible spectrum as the source under consideration. It has been found experimentally that most metals when heated radiate in such a manner that they can be color-matched against a black body. These color matches are very easily and accurately made with an ordinary photometer. It is to be noted that when two bodies have the same color temperature, it is not necessary that they shall have the same brightness for any particular wave-length interval.

To calculate the brightness temperature from readings on the brightness of a non-black radiator, equation 2 is used when the true temperature T is replaced with the brightness temperature S. From this equation and equation 3 the following relation can be derived:

$$\frac{1}{T} - \frac{1}{S} = \frac{\lambda_1 \log \epsilon}{c_2 \log e} \tag{12}$$

In Table XI are given values of T – S for different values of S and different values of ϵ. From the table the correction for any emissive power from $\epsilon = 0.10$ to 1.00 and from any temperature from 1200° K. to 3600° K. can be found. These corrections have been calculated for $\lambda = 0.665\mu$. If a different wave-length is used corrections can be made for the change in wave-length. The same relation holds here as was mentioned above in connection with the correction for the transmission of a window. It was shown there that the correction varied exactly as the variation

TABLE XI.

Corrections to Add to Brightness Temperature Readings for Different Emissive Powers.

Emissive power	Pyrometer using red light, wave-length, $\lambda = 0.665\mu$ at observed temperatures, degree Kelvin of—															
	1000	1100	1200	1300	1400	1500	1600	1700	1800	2000	2200	2400	2600	2800	3000	3600
0.10	119	146	176	209	246	286	329	377	428	542	675	826	998	1193	1413	2246
.20	81	98	118	140	163	189	217	247	279	351	432	523	626	739	865	1322
.30	59	72	86	102	119	137	157	178	201	251	308	371	441	518	603	905
.40	44	54	64	76	89	102	117	132	149	186	227	272	323	378	438	650
.50	33	40	48	57	66	76	87	98	110	137	167	201	237	277	320	471
.60	24	29	35	41	48	55	63	71	80	99	121	145	170	199	299	336
.70	17	20	24	29	33	38	43	49	55	68	83	99	117	136	157	228
.80	10	13	15	18	21	24	27	30	34	42	51	61	72	84	96	139
.90	5	6	7	8	10	11	13	14	16	20	24	28	33	39	45	65
1.00	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

in the wave-length. Thus a one per cent. variation in the wave-length causes a one per cent. variation in the correction. The correction for the transmission of the window through which observations are taken is calculated in exactly the same manner as the correction for the emissive power. From this it is evident that the correction given in the curves in Fig. 5 will give the correction for an emissive power equal to the transmission given in that set of curves.

The Report of the Pyrometry Committee for this year has been made in the form of a summary of existing methods and data. It is planned for next year that the report shall contain current data. It is probable, also, that an attempt will be made to agree upon some definitions and nomenclature.

NELA RESEARCH LABORATORY,
CLEVELAND, OHIO.
July, 1920.

A COÖPERATIVE COLLEGE COURSE IN ILLUMINATING ENGINEERING.

BY

F. E. CADY.

THE proximity of an industrial concern, engaged in the manufacture of electric incandescent lamps and including engineering and physical research departments, to an institution furnishing collegiate technical training, has made possible in Cleveland a coöperative arrangement whereby the institution, with the aid of the industry, has been able to offer a course in illuminating engineering which, since its initiation in 1916, has proved of considerable interest and value.

Shortly after the formation of what are now known as Nela Research Laboratories, Dr. Charles S. Howe, President of Case School of Applied Science, took up with Dr. Edward P. Hyde, the Director of the Laboratories, the question of assisting in outlining and establishing, as a part of the curriculum of the electrical engineering department of the school, a course in illuminating engineering. It was not until the fall of 1916, however, that a comprehensive plan was finally evolved and put into execution, whereby the National Lamp Works of General Electric Company agreed to assist the college with the service of its men and apparatus. Under this arrangement Dr. Hyde laid out the subjects for the lectures and in coöperation with Professor H. B. Dates, head of the electrical engineering department of Case School, has since supervised their presentation, together with accompanying laboratory work.

Three lectures a week, running through two terms, has permitted a choice of subjects which practically covers the whole field of illuminating engineering, as that term is now understood. The theoretical side is presented in the first semester when the student is given a thorough groundwork in the physics of light production, in photometry and in the fundamental principles of illumination, together with light, shade and color. The second term is devoted to the study of the production and characteristics of light sources and illuminating engineering as practiced. In this semester the first lectures are on the manufacture, characteristics and testing of incandescent lamps, arc lamps, vapor lamps

and gas. These are followed by a discussion of glass manufacture and the characteristics and design of globes, shades and reflectors. After showing the relation of illuminating engineering and architecture, specific cases of illumination are presented, starting with general daylight illumination and continuing with the lighting of interiors, such as theatres, museums, churches, schools, libraries, stores, homes, and exteriors, such as streets, parks, etc. A number of lectures are devoted to the subject of projection in all its phases, covering the principles and apparatus, such as searchlights, auto headlights, moving-picture projection, and exterior flood-lighting. Industrial lighting is discussed, together with lighting codes and the complicated subject of rates. It was originally planned to include in the course lectures on the physiology and psychology of vision, but these have not as yet been given. It is expected to provide for them in the near future. A lecture period is assigned to most of the above subjects given in the second term, but some require three or more lectures.

The personnel of the lecturing staff is made up of three men from Nela Research Laboratories, five from the engineering staff of the National Lamp Works, one from the National Carbon Company, one from the Welsbach Company, one from the Cooper-Hewitt Company, one from the Cleveland Illuminating Company, and the architect of the city schools.

When the course was first started each lecturer was requested to prepare a rather full outline of the topics to be discussed and copies of these outlines were placed in the hands of each student. Since then several of the lecturers have written out their material, and arrangements have been made to have all the subjects written in such form that they may ultimately be incorporated in a text-book.

The laboratory work occupies fifteen three-hour periods and is given during the first semester. Twelve experiments have been prepared which are designed to give practice in the use of photometric equipment, familiarity with methods of measurement and practical experience in the use of both instruments and methods. Among the instruments employed are the Lummer-Brodhun contrast photometer, the Ulbricht sphere, different types of flicker photometer, the Sharp-Millar, the Macbeth, and the Beckstein illuminometers as well as such auxilliary apparatus as a mirror

rotator and a universal rotator for measuring distribution curves, and a set of rotating sectored disks for use in connection with the measurement of sources of high intensity. Three of the experiments, which include a study of the characteristics of light sources, are used by the students in solving an assigned problem involving the most economical type and efficiency of lamp to use for a certain installation. Another group of experiments is centred around an illumination survey of the actual lighting conditions in certain rooms. Two members of the staff of Nela Research Laboratories alternate in attendance at the laboratory periods to supervise the experiments and give any assistance that may be desired.

NELA RESEARCH LABORATORIES,
June, 1920.

THE ABSORPTION OF HEAT IN GLASS.*

BY

A. Q. TOOL and C. G. EICHLIN.

INTRODUCTION.

At the annual meeting of the Optical Society, held at Baltimore, December, 1918, a report was given concerning certain phenomena to be observed when glass is subjected to temperatures covering that range usually involved in any process of annealing. Since then more detailed results obtained in a study of these phenomena have been reported in a Bureau of Standards Publication.[1] One of the subjects discussed in these reports was the marked increase in the rate of heat absorption, occurring at a temperature slightly above the usual annealing temperature, while the glass is being heated at a fairly uniform rate. The results obtained for the region in which this absorption occurs were in fair agreement with those published by M. So.[2] This heat absorption was of such a nature that it appeared to be the result of some peculiar transformation in the glass. Since this effect is observed slightly above the usual annealing temperature, it has been used in this laboratory as an aid in determining the upper limit for the temperature region in which glass should be annealed.

Since the publication of the Bureau Scientific Paper mentioned above, a further investigation of this subject has been made, and considerable data have been collected. From these it appears that the treatment to which the glass has been previously subjected affects this heat absorption to a considerable extent. For instance, annealing seems to produce a marked increase in the magnitude of the effect. If this be true, it would appear that this effect is intimately involved in the process of annealing, and should, accordingly, be thoroughly investigated. Furthermore, a careful investigation of the whole subject should be made, because it is sometimes stated in publications dealing with glass [3] that such

* Published by permission of Director, Bureau of Standards.

[1] A. Q. Tool and J. Valasek, *B. S. Sci. Paper* No. 358, Jan., 1920.

[2] M. So, *Proc., Tokyo Math. and Phys. Soc.* (2), **9**, p. 426, 1918.

[3] W. Rosenhain, *Glass Manufacture*, p. 2, 1919.

a heat absorption in glass does not exist. Consequently, it appeared to be of interest to report on the results obtained from a recent series of experiments on a larger number of glasses than were investigated in the previous tests.

Other interesting changes have been observed in the same region, *e.g.*, the increased heat absorption is accompanied by a marked increase in the expansivity, as has been shown by C. G. Peters[4] and C. H. Cragoe. It is also quite likely that the absorption is coincident with a marked change in the specific heat, indications of which have been observed by W. P. White.[5] Furthermore, the change in the temperature coefficient of the refractive index, such as observed by J. O. Reed,[6] is also probably related to this effect. It is possible, further, that the thermal conductivity, and perhaps other physical properties, suffer rather abrupt changes in this region. So far only the change in the expansion has been definitely related to the heat absorption. Further investigation will be required to show whether the other effects occur in exactly the same temperature region, and whether they are so definitely related to the heat absorption as the change in the expansivity.

METHOD.

The observations made in the present series of experiments were confined entirely to the heat absorption. The object was to observe the absorption effect on a number of glasses, using the same definite procedure throughout. This plan furnished more consistent observations on any given sample of glass, a better means of comparing similar effects in different glasses, and greater likelihood of detecting any changes produced in a particular glass by subjecting it to different treatments.

In determining the region of heat absorption the usual differential thermocouple method was used, *i.e.*, the difference in temperature between the glass and a neutral body was observed while both, packed closely together, were being heated at practically the same uniform rate. In this method any endothermic effect occurring in the glass cools it relatively to the neutral body. That is, if the glass while heating is at a slightly higher temperature

[4] C. G. Peters, Meeting Opt. Soc., Baltimore, 1918. C. G. Peters and C. H. Cragoe, JOUR. OPT. SOC., **4**, p. 105, May, 1920.

[5] W. P. White, *Am. J. Sci.* (4), **47**, p. 44, 1919.

[6] J. O. Reed, *Wied. Ann. d. Phys. u. Chem.*, **65**, p. 707, 1898.

than the neutral body, the difference will decrease when this effect occurs, while if it is at a lower temperature, the difference will increase. In case this effect is both of considerable magnitude and occurs in a comparatively narrow temperature range, it is easily detected and the changes in the difference in temperatures may be determined by measuring the differential e.m.f.s between the junctions in the glass and the neutral body.

For measuring the differential e.m.f.s, a Pt. Rh.–Pt.–Pt. Rh. differential thermocouple was used. Ordinarily these e.m.f.s were not measured, but their variations were observed by noting

Fig. 1.

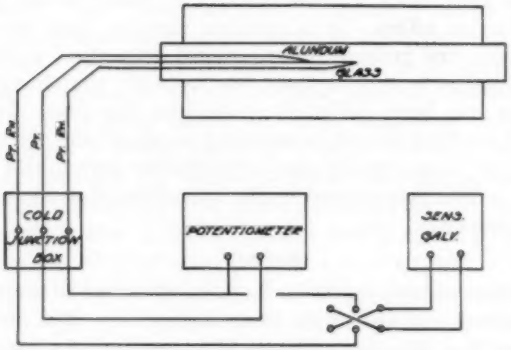


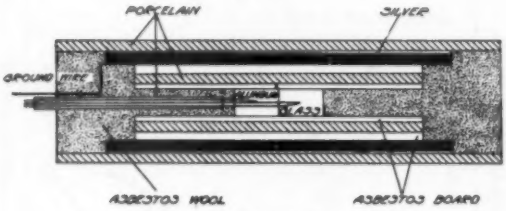
Diagram of apparatus.

the deflections of a sensitive galvanometer in series with the differential thermocouple, as diagrammatically illustrated in Fig. 1. At room temperature a temperature difference of 1° C. produced a deflection of approximately 1.5 cms., at 300° C. of 2.1 cms., at 600° C. of 2.3 cms. In most of the experiments this temperature difference, between the glass and the neutral body, seldom exceeded 5° or 6° C. The deflections of the galvanometer were noted every half minute in the critical ranges, while at other points they were observed less frequently. These deflections were then plotted as ordinates against the temperatures of the glass as abscissæ (see Fig. 3), these temperatures being obtained by determining, by means of a potentiometer, the e.m.f.s between the Pt. Rh. wire of the glass junction and an additional Pt. wire leading to the same junction. The temperatures of the glass were observed

at ten-minute intervals, points between being obtained by interpolation. Measurements involving time were subject to a possible maximum error of about five seconds. There was also the usual error involved in the question as to whether the thermocouple took up the actual temperature of the glass at any instant. If such an error was present, it was presumably the same for all cases where the same rate of heating was used.

Throughout this investigation the mode of packing the glass and the neutral body was the same. (See Fig. 2.) Powdered glass was packed closely about one junction of the differential thermocouple and 120-mesh alundum about the other. These, in close contact, were enclosed in a small porcelain tube, lined with burnt asbestos board and plugged with burnt asbestos wool.

FIG. 2.



Mode of packing.

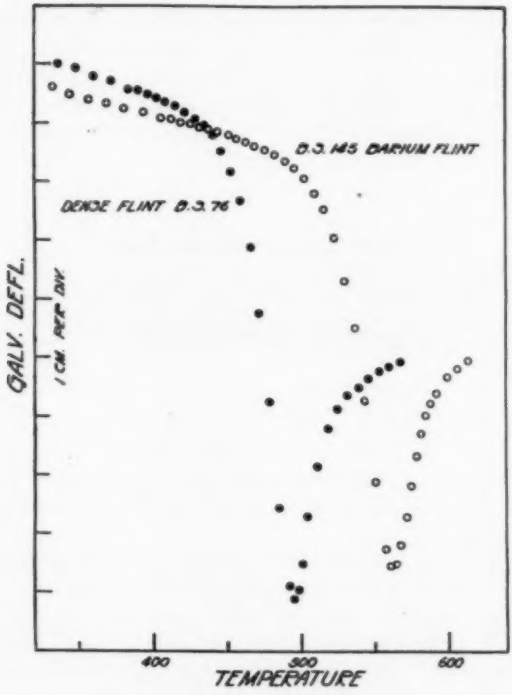
This tube was then placed in a heavy asbestos-lined silver tube. This in turn was packed with burnt asbestos in a closely-fitting porcelain tube which fitted loosely in the tube of the electric furnace.

The glass was powdered in order to facilitate its packing and unpacking. Powdering was especially advantageous, since it made it possible to bring the glass and the junction in close contact, and in certain cases to remove the thermocouple without injury. The quantity of glass used averaged about 1.7 gms. The small porcelain tube, in which the glass and the neutral body were packed, was about 2.5 cms. in length and 3.0 cms. in diameter. Burnt asbestos was used for lining and plugs because it was very convenient for packing and unpacking, and because it appeared to be free from any peculiar thermal properties in the region of the temperatures involved in this investigation. Furthermore, in this region no reactions between the asbestos and the glass, or between the alundum and

the glass, were noticed. The purpose of the silver tube was to equalize, as nearly as possible, the heating of the glass and the neutral body. This tube had a wall thickness of 0.25 cms., a length of 10.2 cms., and a diameter of 2.3 cms.

The heating rate of the platinum-wound electric tube furnace could be controlled quite easily. In most cases an approximately

Fig. 3.



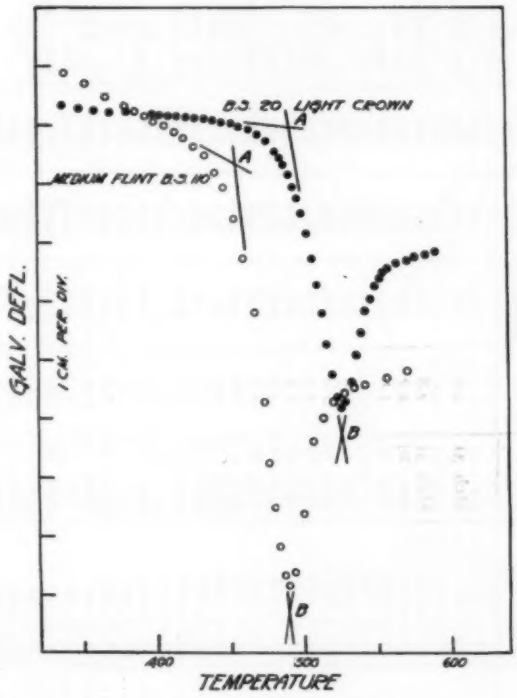
Curves showing heat absorption.

uniform rate of about 6° C. per minute was employed. The heating current was either direct or alternating, as desired. When direct current was used, it was found necessary, owing to leakage, to ground the silver tube, particularly when temperatures much above 600° C. were used. A battery was used whenever a particularly uniform rate of heating was desirable.

DISCUSSION OF EXPERIMENTAL RESULTS.

The accompanying table contains all the data obtained between March 22 and June 7 of this year on the glasses discussed in this article, giving the temperatures obtained for the beginning (A) and the maximum (B) of the cooling effect, when the method

FIG. 4.



Curves showing heat absorption.

shown in Fig. 4 for the determination of these points is used. The data include redeterminations of these points for a number of the optical glasses which were investigated as reported in the previously mentioned Scientific Paper.[7] Determinations on certain other glasses whose expansivity had been measured by

[7] *Loc. cit.*

Table

Name	Source	Date	Mass of glass in grams	Rate of heating in degrees per minute	Critical range in glass		Remarks
					A ±5° C.	B ±5° C.	
Dense flint	B. S. 76	.....	.....	.....	460°	490°	
Dense flint	B. S. 76	Mch. 31	2.6	6.4	460	494	See Fig. 3.
Medium flint	B. S. 110	.....	.....	.....	455°	485°	
Medium flint	B. S. 110	Mch. 31	2.8	6.6	457	489	See Fig. 4.
Medium flint	B. S. 110	Apr. 29	2.2	5.8	452	492	2nd heating.[1]
Medium flint	B. S. 110	Apr. 29	1.8	5.7	459	490	Ann.
Light flint	B. S. 188	.....	.....	.....	485°	525°	
Light flint	B. S. 188	Apr. 2	1.9	7.0	489	527	
Light flint	B. S. 188	Apr. 28	1.8	6.4	490	527	See Fig. 6—M. T. 790°.
Light flint	B. S. 188	Apr. 28	1.8	6.2	487	527	2nd heating.
Light flint	B. S. 188	Mch. 31	2.2	6.7	504	532	Ann.
Light flint	B. S. 188	Apr. 27	1.8	6.0	504	532	Ann.—See Fig. 6—M. T. 780°.
Light flint	B. S. 188	Apr. 27	1.8	6.2	485	525	Ann.—2nd heating.
Light flint	B. S. 188	May 6	2.0	6.2	488	534	Ch. 600°—See Fig. 6.
Light flint	B. S. 188	May 7	1.5	6.7	487	534	Ch. 600°.
Light flint	B. S. 188	May 17	1.8	5.9	485	530	Ch. 900°.
Light flint	B. S. 188	May 17	1.8	5.7	487	531	Ch. 900°.
Barium flint	B. S. 145	.....	.....	.....	520°	560°	
Barium flint	B. S. 145	Apr. 1	2.5	7.0	520	562	See Fig. 3.
Light crown	B. S. 20	.....	.....	.....	495°	525°	
Light crown	B. S. 20	Apr. 1	1.8	6.5	495	526	
Light crown	B. S. 20	May 4	1.1	6.4	492	526	See Fig. 4.
Light crown	B. S. 20	Apr. 30	1.3	6.5	506	534	Ann.—M. T. 675°.
Light crown	B. S. 20	Apr. 30	1.3	6.3	495	529	Ann.—2nd heating.
Boro. crown	B. S. 94	.....	.....	.....	515°	565°	
Boro. crown	B. S. 94	Apr. 1	1.9	6.3	538	572	
Boro. crown	B. S. 94	Apr. 2	1.5	6.3	534	570	
Boro. crown	B. S. 94	May 8	1.8	6.2	538	568	See Fig. 7.
Boro. crown	B. S. 94	May 4	1.2	6.1	540	570	See Fig. 7—Ann.—M. T. 790°.
Boro. crown	B. S. 94	May 5	1.2	6.3	528	568	Ann.—2nd heating.
Boro. crown	B. S. 94	Apr. 7	1.5	6.2	533	576	Ch. 950°—M. T. 710°.

Boro. crown	B. S. 94	Apr. 8	1.5	6.5	534	573	Ch. 950°—2nd heating.
Boro. crown	B. S. 94	May 8	1.4	6.5	530	574	Ch. 950°—M. T. 680°.
Boro. crown	B. S. 94	May 10	1.4	6.6	532	571	Ch. 950°—2nd heating—reground.
Boro. crown	B. S. 94	May 7	1.6	6.5	528	576	Ch. 950°—See Fig. 7.
Light barium crown	B. S. 87	.....	.....	.....	575*	605*	
Light barium crown	B. S. 87	Apr. 2	1.8	6.4	581	620	
Light barium crown	B. S. 87	May 5	1.7	6.5	586	619	See Fig. 5—M. T. 830°.
Light barium crown	B. S. 87	May 6	1.7	6.6	582	617	2nd heating.
Heavy barium crown	B. S. 124	.....	.....	.....	575*	630*	
Heavy barium crown	B. S. 124	Apr. 3	1.8	6.3	612	642	
Heavy barium crown	B. S. 124	Apr. 13	1.9	6.0	610	641	See Fig. 5—M. T. 915°.
Heavy barium crown	B. S. 124	Apr. 13	1.9	6.5	566	604	See Fig. 5—2nd heating.
Plate	Standard	Mch. 27	1.8	6.2	529	564	See No. 4 (Peters & Cragoe).[3]
Plate	Frieden	Mch. 26	2.2	6.6	554	586	See No. 6 (Peters & Cragoe).
Plate	Pittsburg	Mch. 30	1.7	6.0	556	588	See No. 5 (Peters & Cragoe).
Plate	Pittsburg	Mch. 27	1.4	6.2	558	593	See No. 5 (Peters & Cragoe).
Plate	Le Personne	Mch. 24	1.8	6.0	584	614	See No. 7 (Peters & Cragoe).
Therm. Stem	.....	Mch. 29	2.0	6.1	436	474	100° C. scale (H. J. Green).
Therm. Stem	.....	Mch. 23	1.7	6.0	460	494	320° C. scale (Tycos).
Therm. Stem	.....	Mch. 24	2.6	6.0	457	494	320° C. scale (Tycos).
Therm. Stem	.....	Mch. 26	1.7	6.0	558	598	510° C. scale (Tycos).
Therm. Stem	.....	Mch. 30	2.0	6.2	558	595	510° C. scale (Tycos).
Therm. Stem	Jena 59 III	Mch. 25	0.9	5.9	578	609	500° C. scale (Arno Haak).
Therm. Stem	Jena 59 III	Mch. 25	1.9	6.1	574	610	500° C. scale (Arno Haak).
Therm. Stem	Jena 59 III	Mch. 27	1.6	6.8	568	608	560° C. scale (W. Niehls).[3]
Therm. Stem	Jena 59 III	Mch. 30	1.6	6.4	568	606	560° C. scale (W. Niehls).
Therm. Stem	Corning Norm.	Apr. 10	1.3	6.1	536	576	Unico-Beckmann.
Therm. Stem	Corning Norm.	Apr. 14	1.3	6.4	548	584	Unico-Beckmann—M. T. 655°.
Therm. Stem	Corning Norm.	Apr. 15	1.3	6.0	539	571	Unico-Beckmann—2nd heating.
Therm. Stem	Corning Norm.	Apr. 16	1.1	6.3	540	576	Unico-Beckmann—M. T. 825°.
Therm. Stem	Corning Norm.	Apr. 17	1.1	6.5	547	579	Unico-Beckmann—2nd heating.
Therm. Stem	Jena 16 III	Mch. 22	2.4	4.6	542	574	200° C. scale[4]—M. T. 715°.
Therm. Stem	Jena 16 III	Mch. 23	2.4	5.6	542	575	200° C.—2nd heating.
Therm. Stem	Jena 16 III	Mch. 29	2.2	6.4	542	578	200° C.
Therm. Stem	Jena 16 III	Apr. 5	1.5	6.6	541	577	200° C.—See Fig. 8—M. T. 700°
Therm. Stem	Jena 16 III	Apr. 5	1.5	6.3	544	579	200° C.—2nd heating—M. T. 690°.

TABLE—*Continued.*

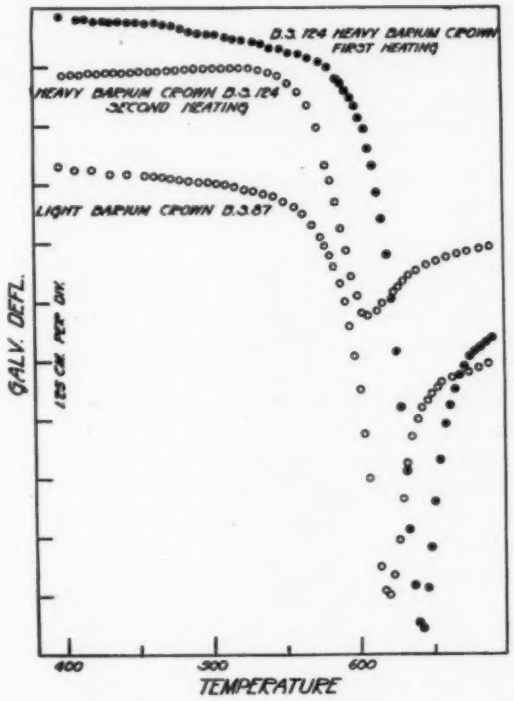
Name	Source	Date	Mass of glass in grams	Rate of heating in degrees per minute	Critical range in glass		Remarks
					A ±5° C.	B ±5° C.	
Therm. Stem	Jena 16 III	Apr. 6	1.5	10.0	547	583	200° C.—3rd heating—See Fig. 8—M. T. 735°.
Therm. Stem	Jena 16 III	Apr. 6	1.5	2.8	537	568	200° C.—4th heating—See Fig. 8—M. T. 655°.
Therm. Stem	Jena 16 III	Apr. 7	1.5	7.1	546	582	200° C.—5th heating.
Tubing	Jena 16 III	Mch. 29	1.7	6.3	544	585	
Pyrex	Tubing	.....	.....	.....	520*	670*	
Pyrex	Beaker	Apr. 8	1.4	6.7	528	671	
Quartz	Cryst.	June 7	1.9	6.4	571	576	See Fig. 9.

* Results given in B. S. Sci. Paper No. 358.
[1] First heating failure—M. T. 505°.
[2] *Loc. cit.*
[3] Peters and Cragoe, Nos. 31 and 32.
[4] Seibert and Kuhn.
Ann.—Annealed at 415° C. for 10 days.
Ch.—Chilled by dropping into water from temperature indicated.
M. T.—Maximum temperature reached.

C. G. Peters and C. H. Cragoe [8] are also included, as well as a few of the glasses often used in thermometer bulbs and stems.

With few exceptions the agreement between the redeterminations and the previous results on the optical glasses was well within experimental error. The discrepancies observed, espe-

FIG. 5.



Curves showing heat absorption.

cially in the cases of the two barium glasses examined, can probably be explained on the basis of the treatment given them before the tests recorded in the previous reports had been made, because it has been found that heating to relatively high temperatures, previous to a test, often lowers the region of the absorption

[8] *Loc. cit.*

effect for comparatively unstable glasses. In the case of the borosilicate crown, the piece of glass from which the samples used in this series of tests were taken, was not the same previously used. This may account, in part at least, for the discrepancies disclosed in this case.

In addition to the temperatures "A" and " B " the table contains the date of the test, the mass of the glass used, the rate of heating, and remarks concerning the source and any special treatment (such as annealing or chilling) to which the glass had been subjected for experimental purposes. Where no such notation is made, the glass had received no treatment other than that usually involved in its manufacture. Whenever the same charge of glass was tested more than once, without repowdering, the succeeding tests are denoted by second, third, etc., heatings. In such cases, the maximum temperature reached during the previous heating is also noted.

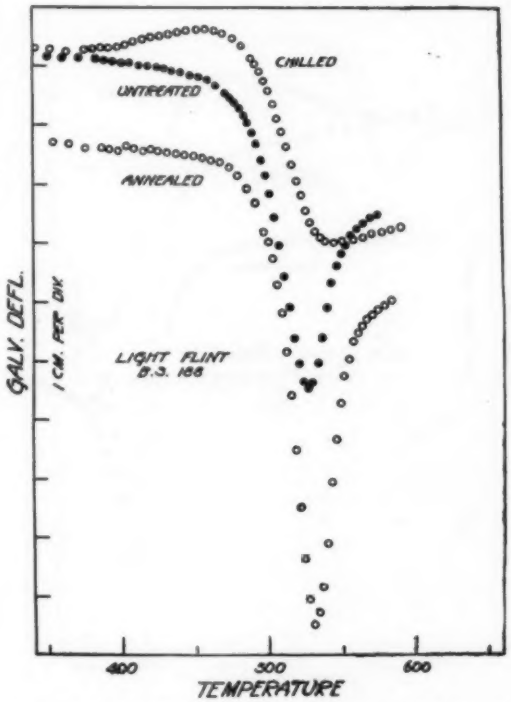
From a comparison of the results gained through a repetition of tests on fresh samples of a glass, it is possible to determine easily the variations to be expected and the accuracy to be obtained in locating this heat absorption region.

Whenever a glass receives only one heating and its temperature is not carried more than 50° C. above the " B " point, and furthermore, if its cooling is reasonably rapid, it is, after the test, still practically in the original powdered form. Besides, if it is a fairly stable glass, the "A" and " B " points for a second heating are shifted very little. Higher maximum temperatures, very slow rates of heating or cooling, or repeated heatings cause the glass particles to fuse into a compact mass, and sometimes, if the glass is unstable, produce considerable changes both in the magnitude of the effect and in the values obtained for "A" and " B." In general, then, when the glass was heated more than once, the effects, as observed in the first heating were not greatly changed in succeeding tests, provided none of the previous heatings had been carried higher than the above-mentioned limit. Such changes as are noticeable consist in sharper returns from the " B " point. (See curves b and c, Fig. 8.) If the temperature is carried to a much higher value on the first heating, the endothermic effect is usually much less pronounced in the succeeding tests. Very often, also, the "A" and " B " points are displaced, usually to lower temperatures. Glasses that show a considerable

tendency to crystallization, especially on the surface, display the greatest shifting of these points. (See Fig. 5.)

Chilling the glass through this region of the heat absorption effect, or annealing it at certain temperatures, also produces marked changes in the character of the curves obtained, as shown

FIG. 6.

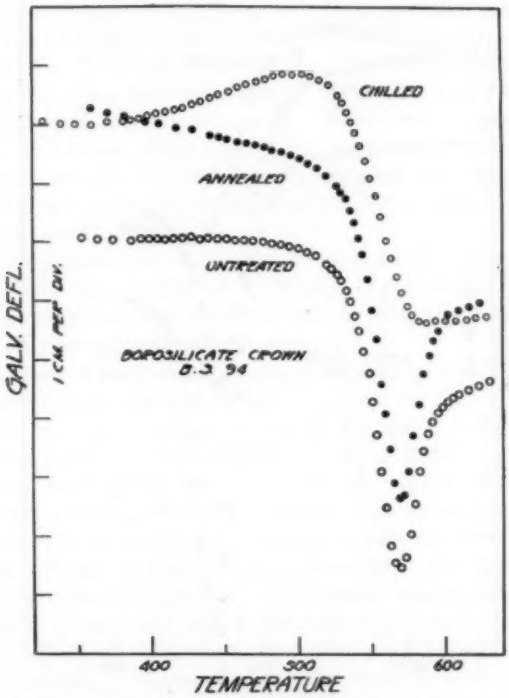


Curves showing effects of chilling and annealing on heat absorption.

in Figs. 6 and 7. In the case of light flint (Fig. 6) the effects, due to chilling, are not so marked as in borosilicate crown (Fig. 7), especially with respect to the rise in the curve (apparently characteristic of chilled glasses) preceding the drop, due to the heat absorption. On the other hand, the effect produced by annealing is more marked in the case of light flint (Fig. 6) than

in borosilicate crown (Fig. 7), probably because the borosilicate crown was annealed at a temperature much lower in relation to the beginning (A) of the heat absorption. Results have been obtained which indicate that annealing for a definite time at temperatures considerably below, or somewhat above "A," pro-

FIG. 7.



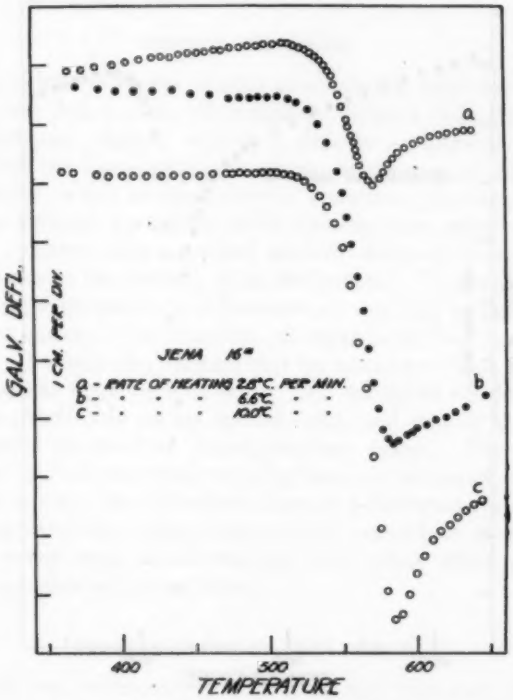
Curves showing effects of chilling and annealing on heat absorption.

duces relatively little or no appreciable change in the form of the curve, but that annealing at some temperature between these extremes will produce a maximum effect. (This question is being further investigated.)

In Fig. 8 three curves are given to show the effects on the position of the "A" and "B" points, produced by changing the

rate of heating. It is evident from these curves that a high rate of heating shifts these points to somewhat higher temperatures, while a slow rate shifts them to lower temperatures. Furthermore, the magnitude of the effect decreases considerably as the rate diminishes. It seems improbable, from observations taken

FIG. 8.



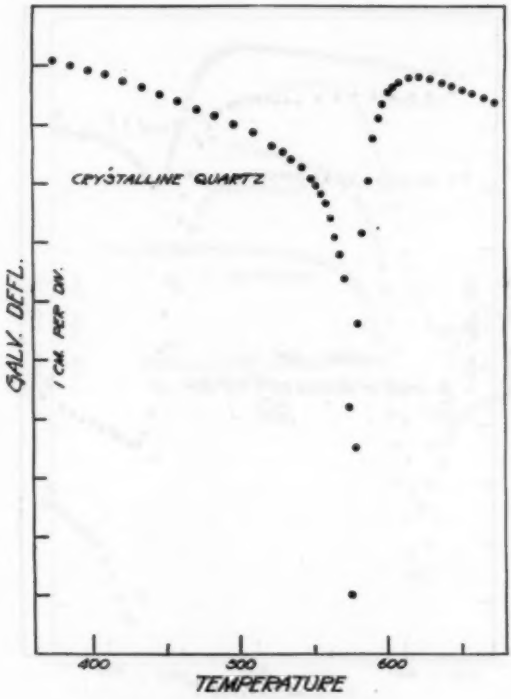
Curves showing effects of change of rate on heat absorption.

by this method on crystalline media, that the shift was due entirely to the failure of the thermocouple junctions to take up the actual temperatures of the glass and the neutral body. This shift in the "A" and "B" points, being small, was not observed in the earlier work, because the curves then obtained were not so well defined as these in the present tests. Consequently, any shifting

in these points, as then observed, was well within the error involved in their determination.

The curves here given, some of which have just been discussed, are chosen as representative of those obtained in the immediate vicinity of the heat absorption. Some show a marked

Fig. 9.



Curve showing heat absorption in quartz.

but uniform slope in the beginning. This is due to the fact that it is practically impossible to obtain, in any case, a horizontal approach to the region of the heat absorption, because of the difficulty in properly adjusting the position of the thermocouple junctions and in controlling the heat flow in the furnace.

The curve in Fig. 9, for crystalline quartz, is added to show

the type of curve that is obtained with this apparatus and method for a transformation in a purely crystalline medium. The difference between the temperatures "A" and "B," determined from this curve for the purpose of comparison, shows what is to be expected when there is little or no lag in the endothermic effect, *i.e.*, where the difference is probably due almost entirely to the temperature gradient in the quartz.

GENERAL DISCUSSION.

It is evident from the results given in the table and shown in the curves that a glass, when heated, shows a heat absorption which increases sharply within a definite temperature range. Throughout the approach to this range, in the case of all glasses so far tested, while in their normal condition, the temperature difference between the neutral body and the glass either remains constant, or shows only a gradual uniform increase or decrease—so far as can be determined from the curves. This would indicate that these temperature differences are due only to the unsymmetrical situation of the junctions, as regards the heat flow in the furnace. It would also indicate that the amount of heat absorbed by the glass at temperatures below "A," indicated on the curves shown, depends only on its specific heat, and that at these low temperatures no marked transformations occur. The relative cooling at "A," characteristic of all glasses so far tested, proceeds with such rapidity that this effect takes on a character very similar to the endothermic effect observed in crystalline media (see Fig. 9), when some transformation takes place which involves a change of state or crystal form.

POSSIBLE CAUSES OF HEAT EFFECTS.

It does not necessarily follow from this fact that glass, or any of its components, has a crystalline structure, and that this structure disappears, or changes form, when the endothermic effect occurs. For there are other possibilities that can be conceived, such, for instance, as the rearrangement or breaking down of certain more or less heterogeneous molecular aggregates which are peculiar to the glass at low temperatures. It is also possible that there may be a certain separation of some of the components on cooling and the dissolving of these on heating.

In this connection there are several facts that seem to differentiate the effects observed in glass from those observed for transformations in pure crystals. For example, this endothermic effect covers relatively a broader range and is considerably shifted towards higher temperatures by an increase in the rate of heating. (See Fig. 8.) Moreover, as was shown in the Scientific Paper previously mentioned, the corresponding exothermic effect on cooling, though marked, is much less distinct, and also shifted more towards lower temperatures, than in the case of those observed in pure crystals.

The question, however, as to whether the structures are crystalline or of some other nature must, so far as the present article is concerned, remain undecided. In this connection, however, it is of interest to consider some of the phenomena which have been recently discussed by Sir Herbert Jackson.[9] According to his discussion it is usually considered that truly vitreous bodies do not fluoresce when subjected to Röntgen rays, ultra-violet light, or cathode rays. Practically all glasses, however, unless chilled very rapidly, do show such fluorescence very distinctly. Such chilled glasses, too, when reheated to a temperature where they begin to deform slightly are again capable of fluorescing. This may be taken to indicate that glass normally cooled actually does possess a crystalline structure. Indeed, in a polycomponent medium, such as glass, only a part of these components need have such a structure. In this respect it would be interesting to determine, if possible, by means of X-rays, whether such structures actually do exist in glass, although a negative result might not conclusively disprove their presence.

If these observed endothermic and exothermic effects in glass are due to the formation of crystalline structures, or some other molecular aggregates peculiar to the glass, it might appear that the breadth of the range and the shifting of the effect with change in the rate of heating, previously mentioned, are due to the lag produced by the high viscosity of the glass. On the other hand, the shifting and breadth of these effects may be due to a superheating or a supercooling, such as is often observed in "slow-melting compounds."[10] It is also possible that the presence of a number of polymorphic crystalline structures, or aggregates,

[9] Jackson, *J. Roy. Soc. of Arts*, **68**, p. 134, 1920.

[10] A. L. Day and R. B. Sosman, *Am. J. Sci.* (4), **31**, p. 341, 1911.

having slightly different melting points, results in a broadening of such an endothermic effect. Some glasses, for example, pyrex, appear, when studied in this way, to have two absorption effects so close together that it is difficult to separate them. In fact, unless care is taken in the observations, the effects appear as one broad region of heat absorption. In any case it would seem reasonable, in view of the phenomena observed where crystallization occurs in undercooled liquids, to assume that the formation of these structures, or aggregates, which produce these effects, begins, on cooling, near the temperature " B "; further, that the tendency for this formation increases continually below this, as the undercooling proceeds, but that the actual power of formation,[11] which was zero above " B," soon reaches a maximum, and again returns practically to zero after still lower temperatures are reached. If such structures, or aggregates, have not been allowed to form while the glass was cooling, it also appears that, on reheating, they will, on reaching a temperature somewhat below "A," begin to form at a rate which increases rapidly as the region of the endothermic effect is approached.

EFFECTS PRODUCED BY CHILLING AND ANNEALING.

In view of these possibilities it would appear that the causes of this endothermic effect may have a far-reaching significance in the problem of annealing. Consequently, it seemed wise to continue the investigation of this endothermic effect and to begin an investigation of the variations produced by annealing and chilling. From the preceding discussion it seems probable that rapid cooling, or even the rates usually employed in annealing, do not allow the formation of the molecular aggregates, whether of a crystalline configuration, or of any other character, which produce the endothermic effect on heating. Preliminary results from investigations of this possibility are given in this paper.

Only two of the glasses mentioned in this article, light flint and borosilicate crown, have been included in the test for the effects produced by chilling. These glasses were heated to temperatures well above " B " and chilled by dropping them into water. In both cases the curves obtained on heating these chilled glasses

[11] A discussion of a very similar condition in the devitrification of glass was recently given by N. L. Bowen, *J. Am. Cer. Soc.*, 2, p. 261, 1919.

indicate an exothermic transformation, occurring below the region of the endothermic effect usually observed in them. Although indications of such an exothermic effect might be expected from the relaxation of stresses in the chilled glass, it seems highly improbable that the effect here observed could be due to this cause, since the glass was powdered. It seems more probable, however, that this exothermic effect has its origin in the same transformation as the exothermic effect observed on cooling, which was shown in the Scientific Paper No. 358. Chilling also reduces the endothermic effect on heating, and renders the point of maximum cooling (B) less well defined, as shown in Figs. 6 and 7. Chilling a glass, therefore, produces an exothermic effect on heating, as shown by these curves, and reduces the normal endothermic effect. It would appear, therefore, that these effects are the result of the formation and dissolution of certain structures whose formation may be suppressed by the rapid cooling.

In the investigation of the effect produced by annealing, four glasses, medium flint, light flint, light crown, and borosilicate crown, were annealed ten days at an average temperature of 415° C. For all these glasses this temperature is well below even the first indication of an endothermic effect. However, it probably falls in a region where the power of formation has not yet become practically zero. It was nearest to point "A" in the case of medium flint, being about 35° below, and farthest from "A," about 115° C. below, for borosilicate crown. In every case there was distinct evidence that this treatment had increased the magnitude of the endothermic effect, although, in practically all instances, the glass had previously received an annealing treatment in the glass factory. For medium flint and borosilicate crown (see Fig. 7), the effect was by no means so marked as for the light crown and the light flint (see Fig. 6). In the latter case the increase in the magnitude of the cooling effect was almost 60 per cent. As may be noted from the table, there was also a tendency to shift the points "B" and "A" to higher temperatures. From these tests it is evident that a prolonged annealing increases the endothermic effect. From this it may be assumed that annealing glass at these temperatures builds up some formation which is peculiar to the glass at these temperatures, and which disappears completely at approximately the temperature "B."

Indications of an increase in the relaxation time [12] after the glass has been held at certain temperatures for a considerable time, have been noted from time to time in this laboratory while viscosity tests were being made. Such increases may also have a bearing on this point. For, if the glass is held for a considerable period of time below the temperature "B," it is possible that the formation of structures, or aggregates, in the glass increases its viscosity.

EFFECT OF POSSIBLE INHOMOGENEITY IN GLASS.

If such a structure can be revealed, however, it becomes extremely important to determine whether it is detrimental to the purposes for which glass is intended. If it is, the problem of annealing then becomes extremely difficult. But it may also be possible that this structure indicates that the glass is in a more stable condition, and may not at all be injurious to it. If this be the case, it would be well in annealing to produce the maximum possible crystalline character of this type.

At any rate it would seem necessary to strive to attain the greatest structural homogeneity in the glass by proper annealing. For instance, it would appear that the heating of the glass to a temperature where it softens considerably, a procedure which is often found necessary in annealing, is required, not so much to reduce large stresses, which would in any case disappear quite rapidly at much lower annealing temperatures, as to bring about greater structural homogeneity in the glass.

Thus there may be a marked difference from point to point in the structure of glass, when chilled or incompletely annealed. Such a condition is evidenced in the Rupert's drop by the increase in the fluorescing power from the surface to its centre. It would appear that much of the peculiar behavior observed in glass not thoroughly annealed may be ascribed to this state of inhomogeneity. For example, when glass is moulded, or worked in any

[12] Relaxation time may be defined as the time required for the viscous or plastic deformation to reach a value equal to the initial elastic deformation, when the viscous or plastic body is being subjected to a constant stress. This ignores the possibility that the rate of deformation immediately after the force has been applied is greater for plastic bodies than at any later time.

way, it usually happens that certain parts of the mass are cooled very rapidly to temperatures below even the region of the heat absorption discussed in this paper, while other parts reach this temperature region much more slowly. As a result there would be, on the basis of the above supposition, a certain inhomogeneity of structure in the glass. Furthermore, since glass is usually annealed at a temperature well below "A," it is possible that some of this inhomogeneity persists, even after the glass has been subjected to the usual annealing procedure.

From the results on the expansivity obtained by C. G. Peters. it would seem reasonable to assume that those portions of the glass which are most vitreous—that is, most free from the aggregates, or structural configurations, that form in the region of the change in the expansivity which he observed—contract more than the more slowly-cooled and less-vitreous parts. This would probably have the effect of causing the glass to contain, on the whole, greater stresses after it had been subjected to any usual annealing and cooling procedure than should normally be expected—this may be the case even though the glass when examined polariscopically, while being held at the annealing temperature, appears to be almost wholly stress-free. Thus, such an effect would probably cause such articles as have received a very sharp chilling on the surface to fail to respond to a treatment that would be sufficient to anneal satisfactorily a more homogeneous sample of the same glass. A more troublesome possibility, however, is the likelihood of a lack of stability, since if the more vitreous parts tend to go over into a more stable form, they will, on the above assumption of their having a greater expansivity, thereby increase in volume. Consequently, the stresses in the glass may continue to grow, withal very slowly, even at room temperature. This possibility would explain on another basis some of the effects discussed by R. Reiger,[13] who, in seeking to explain the failure of Maxwell's relaxation hypothesis in its simple form to apply fully to the phenomena observed in plastic bodies, concludes that one must assume that there are several relaxation phenomena.

This possible change in volume, which must in most cases be very slight, may also explain why some very carefully annealed glasses, considered only from the standpoint of the stresses pres-

[13] R. Reiger, *Verh. d. D. Physik. Ges.*; 21, p. 421, 1919.

ent, warp, or undergo other changes sufficient to render them unfit for certain delicate optical instruments. To be sure, if the structural inhomogeneity disappears at the same rate as the stresses, for the annealing temperature employed, such a possibility could scarcely be considered. However, it is highly improbable that the relaxation of the stresses and the formation of the structural homogeneity, will, at any annealing temperature, occur in even approximately the same time, or that the relaxation time for the stresses and the power of formation of these possible structures in the glass bear identically the same relation to the temperature.

RELATION TO SPECIAL HEAT TREATMENTS.

From the preceding discussion it would seem that the problem of annealing glass for optical instruments, or other delicate apparatus, may involve much more than the mere removal of stresses from glass, and that considerable investigation will be required to show that this removal of the stresses is all that is necessary. On the other hand, for common glassware, or for glass for ordinary optical instruments, it is without doubt sufficient to produce a reasonably stress-free glass. In most cases this can be accomplished very easily by annealing the glass at any temperature within a range of 50° C., or so, below "A" for a reasonable time, depending on the temperature chosen, the viscosity and plasticity of the glass, and the degree of annealing desired, and by cooling it at a rate which is adapted to the rigidity of the glass, and the variation of this rigidity with the temperature.

The lack of stability alone is probably of little importance, if the glass is homogeneous in structure—at least, so far as optical instruments are concerned, since any possible changes in refractive index, etc., due to a slow transformation, are probably small. In thermometers, however, even if the glass is homogeneous, the volume changes due to any instability may be of considerable importance. Thus, it may be that some of the changes observed in the ice-point, and similar phenomena, are due in part to this sort of volume changes, rather than wholly to the stresses. If so, it well may be that a treatment such as that discussed by L. Marchis [14] is valuable in bringing about a con-

[14] L. Marchis, "Les Modifications Permanentes du Verre et le Deplacement du Zero des Thermometres," 1898.

dition of equilibrium. His method consisted of subjecting the thermometer to continued small cyclic changes of temperature at low values—a procedure which he found produced a stabilization much more quickly than holding the thermometer at approximately an equivalent physically constant temperature. He explained the phenomena observed on the basis of the existence in the glass of at least one allotropic or chemical modification. In any case, it would appear that some method of annealing should be followed which would both relieve all stresses and bring about a condition of the greatest possible structural stability.

Another interesting phase of the problem is met with in the so-called "tempered" glass. It is possible that the resisting power of such glasses is not due wholly to stresses set up in them—although it is generally considered that their increased resisting power can be sufficiently explained [15] by these stresses—but to the possible fact that glass in a more vitreous condition is harder and offers more resistance to sudden blows. Such glasses have been studied by various investigators, especially A. de la Bastie [16] and F. Siemens.[17] The results obtained by the latter would seem to support the above view, since he reported that his hardened glass was practically stress-free.

SUMMARY.

Experimental results are given which show that glass on being heated exhibits within a definite and narrow temperature range a marked increase in its heat absorption, which in many ways corresponds to the cooling effect occurring on a change of state.

From a discussion of the possible causes of this effect, it would seem that it may be caused by one or more types of molecular aggregates, which may have a crystalline structure.

It is also shown that chilling or annealing glass produces the effect that should be expected, if the absorption of heat on heating and its evolution on cooling are due to the formation and

[15] Lord Rayleigh, *Phil. Mag.* (6), 1, p. 169, 1901.

[16] A. de la Bastie: *Ann. d. Chim. et Phys.* (5), 23, p. 286, 1881; *Brux. Bll. Phl.*, 14, pp. 118 and 139, 1875.

[17] F. Siemens: *Dtsche Ind. Ztg.*, 26, p. 236; *Chem. C. B.* (3), 16, p. 670; *Nature*, 31, p. 413, 1885.

dissolution of some such crystalline structure or other molecular aggregate.

The possibility that certain methods of cooling or treatment may cause structural inhomogeneity in glass through the non-uniform formation of these aggregates is also discussed, and attention has been called to the importance of this possibility in the annealing and other special heat treatments of glass.

This would mean that annealing is not alone a question of the removing of stresses, but also of producing homogeneity.

BUREAU OF STANDARDS,
WASHINGTON, D. C.

RESONANCE POTENTIALS AND LOW-VOLTAGE ARCS FOR METALS OF THE SECOND GROUP OF THE PERIODIC TABLE.*

BY

F. L. MOHLER, PAUL D. FOOTE and W. F. MEGGERS.

PREVIOUS publications have discussed resonance and ionization potential measurements for five metals of the second group of the periodic table.[1]

The results showed in every case a resonance potential corresponding by the quantum relation $Ve = h\nu$ to the combination series line 1.5 S – $2p_2$ (Paschen notation) and an ionization potential corresponding to the limiting frequency of this series, 1.5 S. Calcium showed in addition a second resonance potential corresponding to the frequency 1.5 S – 2 P, the first line of the principal series of single lines converging at the same frequency 1.5 S.

Spectroscopic studies of low voltage arcs in these metals[2] are not entirely in accord with the work on current-potential measurements.

The glow discharge at potentials below the ionization point shows the line 1.5 S – $2p_2$ in mercury, cadmium and zinc. In cadmium and zinc the line 1.5 S – 2 P also appeared at the potential corresponding to this frequency, but in magnesium the line 1.5 S – $2p_2$ was not observed below the ionization potential.

Davis and Goucher[3] have found evidence of the line 1.5 S – 2P in mercury by photo-electric methods.

The result of the spectroscopic study of magnesium and the failure to observe two resonance potentials in metals other than calcium were points demanding further investigation.

The search for a second resonance potential required a more

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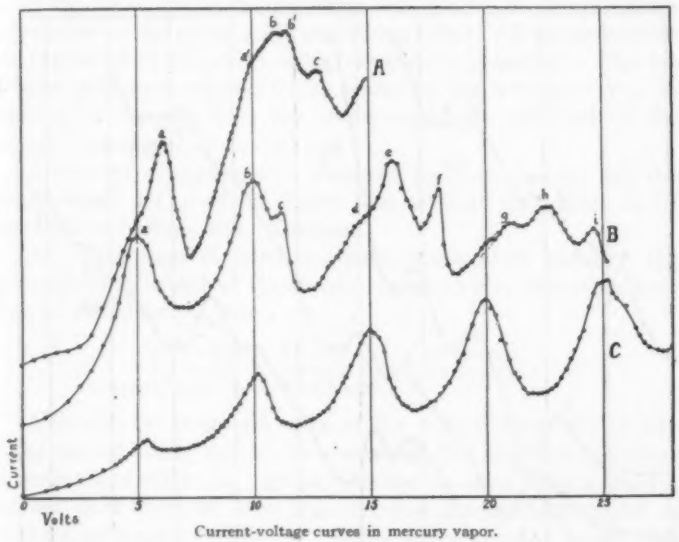
[1] Mercury—Franck and Hertz, *Verh. D. Phys. Ges.* **16**, p. 457, 1914. Also Tate, *Phys. Rev.*, **7**. p. 686, 1916. Cadmium—Tate and Foote, *Bur. Stds. S. P.* 317. Cadmium and Zinc—Tate and Foote, *Phil. Mag.*, 36, p. 64, 1918. Magnesium—Foote and Mohler, *Phil. Mag.*, **37**, p. 33, 1919. Calcium—Mohler, Foote and Stimson, *Bur. Stds. S. P.* 368.

[2] Mercury—Franck and Hertz, *Verh. D. Phys. Ges.*, **16**, p. 512, 1914. Mercury, Zinc and Cadmium—McLennan and Henderson, *Proc. Roy. Soc. A.*, **91**, p. 485, 1915. Magnesium—McLennan, *Proc. Roy. Soc. A*, **92**, p. 374, 1916. Zinc and Cadmium—McLennan and Ireton, *Phil. Mag.*, **36**, p. 461, 1918.

[3] *Phys. Rev.*, **10**, p. 101, 1917.

sensitive means of detecting inelastic impact than is possible with the type of three-electrode vacuum tube previously used. In the present work a four-electrode tube of the type developed for study of non-metallic gases and vapors[4] was employed. The hot wire cathode is surrounded by two cylindrical grids and a plate. The grids, of which one is close to the cathode and the other close to the plate, are maintained at the same potential by metallic contact; a variable accelerating field for electrons is

FIG. 1.



Current-voltage curves in mercury vapor.

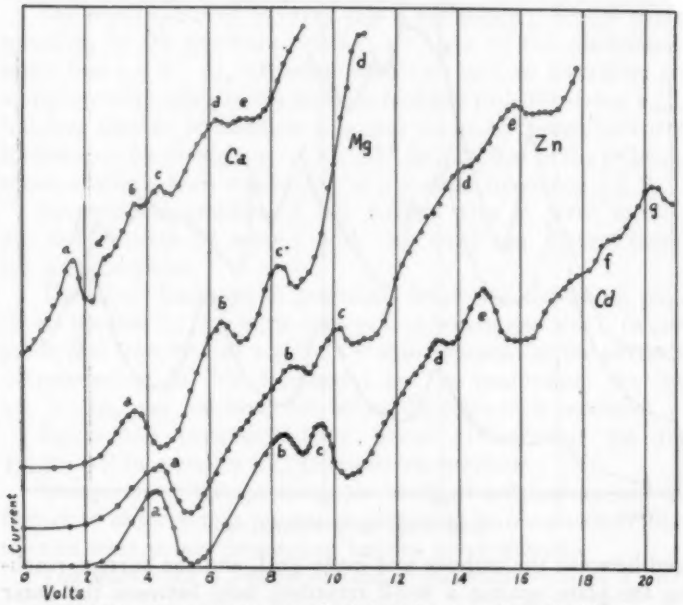
used between the cathode and inner grid, and the current reaching the plate against a small retarding field between the outer grid and plate is measured. The electrodes were contained in pyrex glass tubes in which the metals were boiled.

Fig. 1 shows some typical curves in mercury vapor. Curve A was obtained at low temperature and vapor pressure; curve B at higher pressure; and curve C at still higher pressure and by a method similar to that used in a three-electrode tube. Curve C

[4] Mohler and Foote, *Bur. Stds., S. P.*, in press. Abstract in JOUR. OPT. SOC., 4, p. 49, 1920.

shows inflections only at 4.9-volt intervals due to successive resonance collisions of but one type (the lowest energy exchange on account of a small mean free path). All the inflections in curve B may be explained by successive collisions involving two resonance potentials of 4.9 and 6.7 volts corresponding to the two lines 1.5 S – $2p_2$ and 1.5 S – 2 P. Inflections at points *a*, *b* and *d* are due to successive collisions of the first type; *c* one of the first and one of the second, *e* two of the first and one of the

FIG. 2.



Current-voltage curves in vapors of cadmium, zinc, magnesium and calcium

second, *f* one of the first and two of the second, *g* three of the first and one of the second, *h* two of the first and two of the second, *i* one of the first and three of the second. The curve A shows two additional points. The point *b'* is due to ionization while *a'* gives evidence of a third resonance potential at 8.7 volts. These two points were shown only at low vapor pressure. The failure to observe an inflection arising from one collision of the second type is doubtless due to the overlapping of the effect of the first resonance point. This overlapping of effects limits the precision

possible by this method. Curves of type C give much higher accuracy. Thus the interval between five inflections is $19.5 \pm .2$ volts, giving for the observed first resonance potential of mercury $4.88 \pm .05$ volts while the theoretical value is 4.865 volts.

Fig. 2 illustrates one curve for each of the metals cadmium, zinc, magnesium and calcium. The latter, taken from the work by Mohler, Foote and Stimson,[5] was obtained by use of a three-electrode tube. The curve for cadmium is very similar to the curve B for mercury; the curves for zinc and magnesium less so as the vapor pressure obtainable was limited to lower values on account of the pyrex glass containing tubes. All the inflections are explained by the effect of two resonance potentials. The inflection points correspond to the points on the mercury curve B. Only with calcium does the *single* resonance collision of the second type appear, indicated at *a'*.

A number of curves were obtained for each element and the mean values are given in Table I as well as the values to be expected in barium and strontium.

As to the possible third resonance potential of mercury, the second lines of either of the absorption series give potentials very close to the observed point.

$1.5\ S-3\ p_1$	$\lambda = 1435.6$	$V = 8.60$ volts
$1.5\ S-3\ P$	$\lambda = 1402.7$	$V = 8.80$ volts
Observed mean value	$V = 8.7$ volts	

Although the observed value of 8.7 volts corresponds closely to the second line of either of these series, and although collisions of such types might be expected on the basis of Bohr's theory, further work must be done from the spectroscopic standpoint to substantiate such a conclusion. Foote and Meggers [6] found that collisions corresponding to the second member of a series did not exist in vapors of the alkali metals, but the alkali earth metals may, of course, behave differently.

The similar characteristics of electronic impact in vapors of all the metals of group II of the periodic table require that their spectra in low-voltage arcs should be analogous. Hence the single-line spectrum of each of these metals should be $1.5\ S - 2p_2$. As McLennan [7] observed $1.5\ S - 2\ P$ for the single-line spectrum

[5] *Loc. cit.*

[6] Foote and Meggers, *Phil. Mag.*, **40**, p. 80, 1920.

[7] *Loc. cit.*

TABLE I.

Ionization and Resonance Potentials.

Metal	Series Paschen notation	Wave length A in vac.	Computed potential	Observed potential
Zinc	1.5 S (a)	1319.98	9.3524	9.3
	1.5 S−2p$_2$	3076.88	4.0122	4.18
	1.5 S−2P	2139.33	5.7705	5.65
Cadmium	1.5 S (a)	1378.69	8.9542	9.0
	1.5 S−2p$_2$	3262.09	3.7844	3.95
	1.5 S−2P	2288.79	5.3937	5.35
Mercury	1.5 S (a)	1187.96	10.3918	10.2
	1.5 S−2p$_2$	2537.48	4.8651	4.76
	1.5 S−2P	1849.60	6.6744	6.45
Magnesium	1.5 S (a)	1621.72	7.6123	8.0
	1.5 S−2p$_2$	4572.65	2.6997	2.65
	1.5 S−2P	2853.06	4.3269	4.42
Calcium	1.5 S (b)	2028.20	6.0867	6.01(c)
	1.5 S−2p$_2$	6574.59	1.8777	1.90
	1.5 S−2P	4227.91	2.9198	2.85
Strontium	1.5 S (b)	2176.94	5.6708	
	1.5 S−2p$_2$	6894.45	1.7906	
	1.5 S−2P	4608.61	2.6787	
Barium	1.5 S (b)	2379.28	5.1885	
	1.5 S−2p$_2$	7913.52	1.5600	
	1.5 S−2P	5537.04	2.2295	

[a] Dunz Tubingen dissertation.
[b] Data furnished by F. A. Saunders.
[c] Mohler, Foote and Stimson.

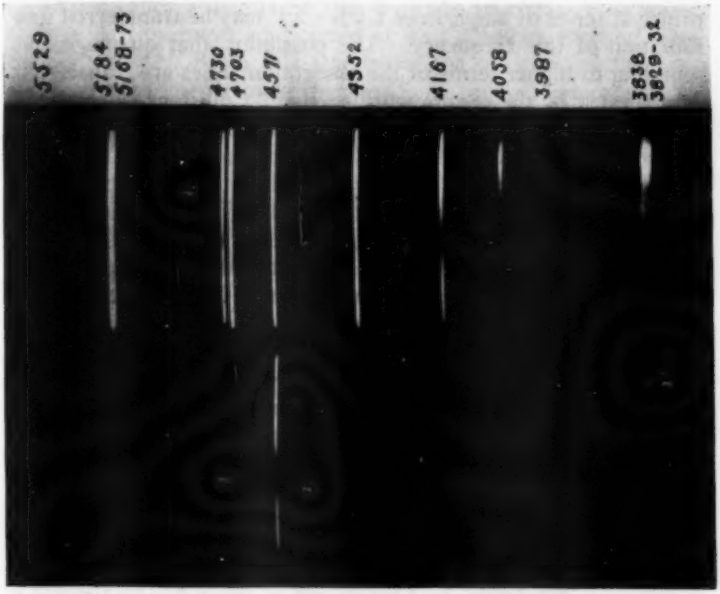
of magnesium and could not detect the presence of 1.5 S − 2p$_2$ below the ionization potential, it was desirable to repeat his experiment. The evidence that ionization is unnecessary to produce the line 1.5 S − 2 P ($\lambda = 2852$) is conclusive, so that the present experiment was concerned with determining whether or not the line 1.5 S − 2p$_2$ ($\lambda = 4571$) is excited below the ionization potential.

A cylindrical anode and a hot-wire cathode along its axis were placed at one end of a long tube with a glass window sealed on the other end. Only the part of the tube near the electrodes was heated. The light of the incandescent cathode was shielded from the window so that the glow discharge appeared against a dark background. This was photographed by a single glass prism Zeiss spectograph of large aperture. Fig. 3 shows the result. The

upper spectrum is the arc at 14 volts, the lower at 3.5 volts. The former required a 5-minute exposure, the latter about 4 hours. The current ratio for the two cases was about 50 to 1.

The reproduction masks the relative intensity of the 14-volt arc lines. The lines on either side of $\lambda = 4571$ are much more intense than this line, so that the transition to the single-line spec-

FIG. 3.



Spectra of magnesium arc. Upper 14 volts: Lower 3.5 volts showing the single line spectrum $\lambda = 4571$.

trum is most marked. The photograph shows conclusively the existence of 1.5 $S - 2p_2$ ($\lambda = 4571$) as the single-line spectrum of magnesium.

It is interesting to note that the 5-minute exposure at 14 volts shows all the arc lines listed in Kayser and Runge's tables in this region. The lines belong to the triplet series and single series. The first lines of the first and second subordinate series of triplets and six lines of the first subordinate series of single lines are shown in the reproduction.

We are accordingly able to state the following general laws for the phenomena of electron impacts with molecules of metals of the second group. There are at least three types of inelastic impact, two resonance potentials, and an ionization potential. (1) Electrons with kinetic energy exceeding one quantum of the frequency of the combination line 1.5 S – $2p_2$ may lose this amount of energy in exciting radiation of this single frequency. (2) Likewise a quantum of the frequency of the first line of the principal series of single lines 1.5 S – 2 P may be transferred into radiation of this frequency. The possibility that quanta corresponding to higher terms of the absorption series are transferred is suggested by the results with mercury. Collisions of this type must be very improbable and we feel that the evidence here given is insufficient for deciding this important point. (3) Electrons with energy greater than that corresponding to the common limit of the two absorption series, 1.5 S, may ionize the molecule. While the phenomenon of successive inflections corresponding to successive *resonance* collisions shows that in this case the transfer of energy must take place by quanta, there is no corresponding evidence for ionization collisions. It is entirely in accord with atomic theories that this latter energy transfer should not be by quanta. Thus, when an electron with velocity exceeding the ionization potential ejects an electron from an atom, it may divide its remaining kinetic energy with the ejected electron so that the greater its energy before collision the greater will be its loss of energy. The many-line spectrum emitted above the ionization potential is probably to be ascribed to recombination of ionized atoms with electrons. This spectrum consists of the single-line and triplet series, as shown in the 14-volt arc spectrum of magnesium. In addition to the arc spectrum obtained under the condition of our experiment there are a large number of enhanced spark lines, including a set of doublet series. This fact would indicate that there exists for these elements a second ionization potential corresponding to the formation of doubly-charged ions which on recombination give the latter spectrum. It may be possible to observe this second type of ionization by the methods here employed. If so, considerable information would be available in regard to enhanced spectra.

BUREAU OF STANDARDS,
August 5, 1920.

THE PRECISION OF PHOTOMETRIC MEASUREMENTS.

BY

F. K. RICHTMYER and E. C. CRITTENDEN.

THE extent to which the advance of Physical Science is dependent on our ability to make more and more precise measurements was discussed most admirably by Vice-President Zeleny in his retiring address before Section B of the American Association for the Advancement of Science at its Columbus meeting, December, 1915.[1] From this standpoint the question of the precision of photometric measurements is of peculiar importance in that in this field, more than in any other, the precision obtainable is limited by other than physical factors; namely, by the ability of the eye to decide when two adjacent areas appear equally bright. This physiological factor seems to set a definite limit to the precision attainable in photometry. We may improve the constancy of our light sources; make better photometer tracks; provide more comfortable and more favorable conditions under which the eye works, such as a head rest, stationary photometer, exclusion of side lights, a "good" photometric field, absence of noise and vibration, most favorable field brightness, etc., but we cannot go beyond the limits set by the sensibility of the zero instrument used, the eye. What these limits are, and whether they have already been reached, are questions of much importance not only to the practical photometrist, but to the investigator who is using photometric methods in pushing forward the boundaries of science.

During the summer of 1915 the writers, in collaboration with a number of observers at the Bureau of Standards, collected a large amount of photometric data,[2] under conditions which called for the highest available precision. Although not contemplated at the time, it has since seemed desirable to examine these observations to ascertain what precision obtains in photometric work. For the purpose of such an analysis, the material available is somewhat unsatisfactory in that it was intended for an entirely different purpose, and therefore leaves untouched a large number

[1] *Science*, Vol. xliii, p. 185, 1916.

[2] See paper by Crittenden and Richtmyer. "An 'Average Eye' for Heterochromatic Photometry, and a Comparison of a Flicker with an Equality of Brightness Photometer." *Trans. Ill. Eng. Soc.*, Vol. xi, p. 331, 1916, and *Bureau of Standards Scientific Paper* No. 299.

of questions of fundamental importance. The results, however, are regarded as of special significance because of the fact that the observers, in making the settings, were not conscious of being studied for the precision of which each was capable. It is thought that actual working conditions have been studied, and that the conclusions, though far from being exhaustive, are free from any disturbing factors, more or less psychological, which must inevitably be present in any work deliberately intended for studying precision.

Precision photometry, as we understand the term to-day, was made possible by the development of the incandescent lamp and the storage battery, and began with the invention of the now well-known Lummer-Brodhun photometer in 1889. Previous to this time a great deal of attention had been given to studying the sensitiveness and theory of operation of the Bunsen screen, notably by Weber,[3] who found the "mean error" of a single observation with the Bunsen photometer to vary from 1.8 to 4.7 per cent. It was later pointed out by Lummer and Brodhun that a part of this variation was undoubtedly due to the fact that Weber used open gas flames. Later observers, of course, used incandescent lamps.

In the paper describing their new photometer,[4] Lummer and Brodhun discuss the theory of the Bunsen screen, and, applying the same theory to their device, show that, for the same sharpness of division between the adjacent fields, their photometer should be about 2.5 times as sensitive as the Bunsen. Actually, they state, the ratio is somewhat larger because of the fact that the dividing line in the Lummer-Brodhun field is somewhat sharper than in the Bunsen field. Lummer and Brodhun state that for their photometer they found the "mean error" of a single setting to be under 0.5 per cent., and that a difference in brightness of 1.5 per cent. between the two fields was easily observable. (These measurements refer to the so-called "equality" field; not to the contrast pattern.) The field size and field brightness used are not given except to state that they were, so far as brightness is concerned, working in the region of maximum sensibility.

Later, in the description of their contrast photometer,[5] which

[3] *Ann. d. Ph.*, Vol. xxxi, p. 676 (1887).
[4] *Zeit. fur Inst.*, Vol. ix, p. 41 (Feb., 1889).
[5] *Zeit. fur Inst.*, Vol. ix, 461 (Dec., 1889).

they show ought to be approximately twice as sensitive as the "equality" photometer, they find for the ordinary contrast cube with stationary absorbing glasses (*i.e.*, 8 per cent. contrast) a "mean error" of one setting in a series of 20 to be 0.40 per cent. Maximum sensibility was found with the 3.5 per cent. contrast. In this case the mean error of a single setting was 0.22 per cent.

Martens and Bechstein [6] also compared the contrast type with the equality as well as several other forms of photometer, and reported the "mean errors" as follows:

Photometer	Mean error
	Per cent.
Lummer-Brodhun contrast	0.38
Martens	0.46
Lummer-Brodhun equality	0.59
Joly diffusion	1.7
Bunsen-Rudorff grease-spot	2.0

In a paper on "The Sensitiveness of Photometers" [7] L. W. Wild reports an examination of the sensibility of a number of different types of photometers. His method was as follows: "The photometer box was moved slowly out of balance to the right, and was then brought back slowly till the balance reappeared, the reading being taken." A similar reading was then taken to the left. "The difference between these two readings (left and right) is a measure of the sensitiveness of the photometer." Each pair of readings was taken six times and the mean of the six used. The following table gives the results:

Photometer	4.32 wpc. carbon against 4.35 wpc. carbon illumination = 1.2 foot candles	4.3 wpc. carbon against 1.4 wpc. osram illumination = 1.65 foot candles	Pentane lamp against incandescent gas mantle
	Per cent.		
Ritchie wedge	2.4	5.0	8.2
Joly paraffin prism	2.5	4.8	8.3
Lummer-Brodhun (equality?)	0.7	3.5	8.0
Bunsen, ordinary single	1.2	3.6	8.7
Bunsen, ordinary double	1.5	5.0	9.8
Bunsen, special single	0.4	3.3	7.8
Bunsen, special double	1.0	4.5	9.0
Simmance-Abady flicker	1.7	1.8	2.1
Wild flicker	0.8	0.8	1.0

[6] *Jour. f. Gasbeleuchtung*, Vol. xliii, p. 251 (1900).

[7] *Electrician* (London), Vol. lx, p. 122 (Nov., 1907).

To show the effect of even a small color difference in the field the sensibility of the Special Bunsen (single) was determined for the relativity small color difference: 4.32 wpc carbon against 3.85 wpc carbon. The value reported is 2.6 per cent., showing that the photometer is less than one-fourth as sensitive as with a color match.

Kennelly and Whiting [8] studied four different photometers, determining the probable error of a single observation for a color match (0.24 mean spherical candle per watt carbon) and for the color difference given by 0.19 mean spherical candle per watt carbon against 0.55 mean spherical candle per watt tantalum. The results for a color match were as follows:

Photometer	Probable error of a single observation
	Per cent.
Lummer-Brodhun contrast	0.5
Lummer-Brodhun equality	0.75
Lummer-Brodhun concentric	1.0
Bunsen (grease-spot)	1.5

In the heterochromatic measurements the probable error of a single observation varied from 1.4 to 1.9 per cent. With a color match the Lummer-Brodhun contrast photometer was three times as sensitive as the Bunsen. But with a color difference the two photometers had more nearly the same sensibility. It is further stated that the constant errors (errors in mean results) obtained by different individuals averaged only one-third the probable error of a single observation in the case of the homochromatic settings, but *twice* the probable error in the heterochromatic.

In the discussion of the paper Mr. J. B. Klumpp [9] thought that the precision stated for the Bunsen was too low in comparison with the Lummer-Brodhun; that the precision with experienced operators should be from one-fourth to one-half per cent. Mr. P. S. Millar reported experiments with the Bunsen (Leeson disk) and with the Lummer-Brodhun contrast photometer. In a series of 25 settings the former gave a probable error of 0.4 per cent., and the latter of 0.24 per cent.

[8] Proc. Natl. Elect. Light Assn., 1908. Abstract in *Elect. World*, Vol. li, p. 1104 (1908).

[9] *Elect. World*, Vol. li, p. 1152 (1908).

Later, Edwards and Harrison [10] reported an analysis of some 5000 readings to test the relative accuracy of the Bunsen and the Lummer-Brodhun low-contrast type. They find the average deviation of a single setting to be 0.4 per cent. for the Lummer-Brodhun, and 1.5 per cent. for the Bunsen. They further state that *no marked loss of precision results from small color differences such as with a carbon against a tungsten* incandescent lamp. This conclusion is quite contrary to that given by Wild and by Kennelly and Whiting.

Separating out from the above the results on the Lummer-Brodhun we find the following, for color match:

TABLE I.

Observer	Type	Precision of single observation
Lummer and Brodhun	Equality	" Mean error " (less than) 0.5 per cent.
Lummer and Brodhun	Contrast (8 per cent.)	" Mean error " 0.4 per cent.
Lummer and Brodhun	Contrast (3.5 per cent.)	" Mean error " 0.22 per cent.
Martens and Bechstein	Equality	" Mean error " 0.59 per cent.
Martens and Bechstein	Contrast (high?)	" Mean error " 0.38 per cent.
Wild	Not given	"Sensitivity " 0.7 per cent.
Kennelly and Whiting	Contrast (high?)	Probable error 0.5 per cent.
Kennelly and Whiting	Equality	Probable error 0.75 per cent.
Kennelly and Whiting	Concentric	Probable error 1.0 per cent.
Millar	Contrast (high?)	Probable error 0.24 per cent.
Edwards and Harrison	Contrast (low)	Average deviation 0.4 per ct.

It is rather difficult to reduce these errors to a common denominator. " Mean error," as used by Lummer and Brodhun and by Martens and Bechstein, is not clearly defined. It probably means " average deviation," *i.e.*, the average of the deviations of the several observations from their mean, rather than " mean error " which is seldom used and which is the square root of the mean square deviation. " Sensitivity " as measured, probably gives the average *maximum* departure from the mean. Probable error, E, and average deviation, a.d., are connected by the approximate equation,

$$a.d = 1.2\ E$$

Assigning equal weights to the four values for the 8 per cent. (high) contrast field given above, one obtains 0.4 per cent. as the *average deviation* of a single observation from the mean. The validity of this average is doubtful, for the four values were probably obtained under different conditions as regards field brightness, field size, etc.

[10] *Trans. Ill. Eng. Soc.*, Vol viii, p. 633 (1913).

With regard to the effect of brightness and size of field on the precision of observations very few definite measurements have been published. König and Brodhun's[11] fundamental data on the least perceptible brightness difference apparently show that no important change in this quantity occurs in the range of brightness which corresponds to illuminations of 1 to 100 foot-candles on the photometer. Uppenborn,[12] however, made experiments with two observers which showed a maximum precision of setting with an illumination of about 30 meter-candles on the photometer. With this illumination the "mean error" of an observation was found to be about 0.5 per cent., while at 5 and at 50 meter-candles it was about 1.0 per cent. The type of photometer is not mentioned.

Uppenborn's observations were made with an exact color match. When color differences are involved in the measurements, an increase in the illumination in general increases the difficulty of equating the brightness of the two parts of the photometric field and tends to reduce the precision. Such measurements can therefore be made more easily and precisely at relatively low illuminations, but reliable quantitative data are not at hand to show how much is gained in precision in any particular case by reduction in brightness of the field.

The present paper is the result of an analysis of nearly 1500 sets of observations of from 10 to 20 settings each (approximately 20,000 settings in all) by 15 different observers. Some of these were experienced photometrists, and all were accustomed to making careful observations in various kinds of laboratory work. They were selected from the much larger group which took part in the work, not because of any peculiarities in their observations, but because they happened to have made the various settings which it was desired to analyze. For simplicity we shall designate the several observers by the letters B, C, D, F, G, H, J, K, M, N, R, S, T, V, and W, respectively.

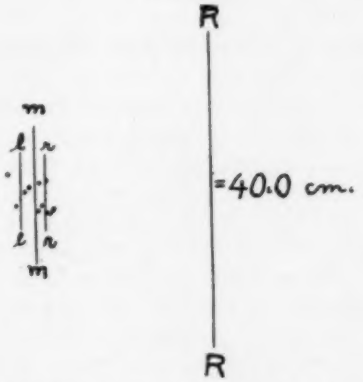
The apparatus and the method of taking and recording the observations have already been described by the writers in the paper referred to. In all the observations, the left lamp and the photometer were kept stationary. The lamp at the right was

[11] Sitzungsber, Akad. Wiss. Berlin, 1888, p. 917.

[12] Uppenborn-Monasch Lehrbuch der Photometrie, R. Oldenbourg, Munich and Berlin, 1912, p, 212.

moved by turning a wheel beneath the photometer. The settings of this lamp were printed on a sheet of paper fastened to a drum which was placed immediately beneath the carriage, and which, after each observation, was rotated automatically a slight amount so as to prevent one "record" falling on top of a preceding one. The distance of the lamp from the photometer is easily determined by measuring from reference lines on the paper. The photometer was provided with an eye-shield, which also served as a head-rest, and in some cases outside light was further excluded by a cloth thrown over the head. The observer was seated on a stool, the

FIG. 1.



height of which could be adjusted so as to allow a comfortable position while observing.

Fig. 1 represents a set of 10 readings. RR is the reference line, which in this case was 40.0 cm. from the photometer. The line mm was located "by eye" to represent the average of the 10 readings. Its distance from the photometer can at once be found. In this set it was 36.81 cm. (Actual tests showed that in this graphical method of locating the mean of a number of observations, unless the observations were very badly scattered, an error of more than one- or two-tenths of a millimetre was seldom made.) It is seen that four of the observations fall on the left of the mean, and six on the right. The line ll is the mean position of the points on the left, and rr the mean of the points on the right. The distance ml gives the average deviation to the left, and likewise mr to the right. In this set ml = 0.23 cm.

and mr = 0.15 cm. The average distance in cm. of the several observations from mm is

$$\text{Aver. dist.} = \frac{4 \times 0.23 + 6 \times 0.15}{10} = 0.18 \text{ cm.}$$

The average deviation of a single observation from the mean, expressed in per cent., is

$$\text{a.d.} = 100 \times \frac{2 \times 0.18}{36.81} = 0.98\%$$

This graphical method of determining average deviation has been used throughout. In like manner the maximum deviation of any single observation from the mean has been determined. This latter value should correspond approximately to the "sensitivity" as determined by Wild.

PRECISION IN HOMOCHROMATIC PHOTOMETRY.

These measurements with a color match were made with the Lummer-Brodhun 8 per cent. contrast photometer, with three different field colors: (1) 4 wpc carbon; (2) a blue resulting from filtering 4 wpc. carbon light through 1 cm. of a copper sulphate solution containing 53 grams per litre; (3) a yellow resulting from filtering 4 wpc carbon light through 1 cm. of a potassium bichromate solution containing 72 grams per litre. (These are the so-called blue and yellow solutions in the paper above referred to.) Although the contrast field was used, the observers in making the settings directed their attention not primarily to the contrast trapezoids, but to the narrow strips between the trapezoids, practically using therefore the equality method of setting, rather than the contrast. The following table gives the results:

TABLE II.

Color	Observer	Number of sets examined (10 to 20 settings each)	Average a.d. of a single observation	Average maximum deviation
Carbon.....	C	25	0.301	0.780
Carbon.....	R	44	.310	.828
Carbon.....	T	97	.314	.825
		Mean	0.308 per cent.	0.811 per cent.
Blue.....	C	67	.327	.830
Blue.....	R	138	.318	.780
Blue.....	T	74	.299	.723
		Mean	.315 per cent.	.778 per cent.
Yellow.....	C	40	.306	.824
Yellow.....	R	33	.342	.912
Yellow.....	T	71	.309	.813
		Mean	.319 per cent.	.850 per cent.
		Mean of all	.314 per cent.	.813 per cent.

The maximum deviation from the mean is about 2.5 times the average deviation from the mean. These values for average deviation seem to be sufficiently consistent to warrant the statement, that, with the photometer used, a careful observer, accustomed to the apparatus and taking data under favorable conditions, should maintain an average deviation of a single observation from the mean of its set of about 0.3 per cent. In a set of 25 observations the average deviation of the mean (A.D.), *so far as errors of setting are concerned,* should be about 0.06 per cent. It is interesting to note that this is the order of magnitude of the variation to be expected between the mean of sets if the relative voltage of the lamps is maintained constant to .01 per cent. In other words, if errors in photometric measurements of this kind, due to variation in voltage, are to be made small in comparison with the errors due to the photometric settings themselves, the voltage must be maintained accurate to the order of .001 per cent.

PRECISION WITH A SLIGHT COLOR DIFFERENCE.

In the course of measurements on the temperature coefficient of transmission of the two solutions, it was noticed that the potassium bichromate solution became distinctly redder with rising temperature. The measurements were made by placing one cell at 20° C. on one side of the photometer, and the other at temperatures of 25°, 30°, and 35°, successively, on the other side. For the temperature difference 20°–25°, the color difference in the two photometer fields was just barely perceptible. In the case of the temperatures 20° – 30° the difference in color was easily observed, while for the temperatures 20° – 35° it was very marked, although not nearly so great as the difference between a carbon and a tungsten lamp. The measurements were made by eight different observers, some of whom were not accustomed either to the apparatus or to the method of setting, and consequently the precision was somewhat lower. The results are given in Table III.

There appears to be no marked loss of precision arising from a *slight* color difference, since the first of these three values of average deviation, where there is practically no color difference, is almost exactly the average of the other two.

TABLE III.

Photometer: Lummer-Brodhun Contrast.

Color difference (in terms of temp. difference)	Observers	Total number of sets examined	Average a.d. of one setting	Average maximum deviation
			Per cent.	*Per cent.*
20°–25°	C, F, G, H, K, R, S, T.	23	0.393	1.25
20°–30°	C, F, G, H, K, R, S, T.	21	.386	1.04
20°–35°	C, F, G, H, K, R, S, T.	24	.404	1.11

PRECISION IN HETEROCHROMATIC PHOTOMETRY WITH LUMMER-BRODHUN CONTRAST PHOTOMETER.

Two color differences, most commonly met in practice, were studied: (1) 4 wpc carbon against 1.2 wpc tungsten, and (2) 4 wpc carbon against 0.65 wpc gas-filled tungsten. Ten observers took part in the tests. These will be divided into two groups: Group A, containing three observers, C, R, and T, whose work was analyzed in Table II, and who had had a comparatively large amount of experience in making settings with a color difference; and Group P, containing seven observers, only three of whom had had experience even in *homochromatic* measurements. The results are shown in Table IV. For the sake of comparison the average deviation obtained by these groups with a color match (4 wpc carbon) is shown. In all these measurements the effective field brightness was approximately 2.5 millilamberts. (Effective illumination, 25 meter-candles.)

TABLE IV.

Photometer: Lummer-Brodhun Contrast.

Color difference	Observers	Total number of sets examined	Average a.d. of one setting	Average maximum deviation
			Per cent.	*Per cent.*
None:	Group A:			
Carbon-carbon.....	C, R, T.....	18	0.345	0.78
Carbon-carbon.....	Group P: B, F, G, J, K, S, W.....	42	.58	1.60
Carbon vs. tungsten.	Group A.....	6	.60	1.95
Carbon vs. tungsten.	Group P.....	14	1.36	3.40
Carbon vs. gas-filled.	Group A.....	18	1.00	3.45
Carbon vs. gas-filled.	Group P.....	42	1.95	5.20

The progressive decrease in precision, with increasing color difference, is obvious. The proportionate change from color match to the color difference presented by a carbon and a gas-filled tungsten lamp is about the same for Group A as for Group P. Approximately it may be said that with the carbon-tungsten color difference the precision is one-half as great, and with the carbon-gas-filled one-third as great as with a color match.

An attempt was made to measure the transmission of the yellow solution and of the blue solution, mentioned above, by the Lummer-Brodhun photometer. Even with experienced observers the precision was so low that the experiments were abandoned.

PRECISION IN HETEROCHROMATIC PHOTOMETRY WITH THE FLICKER PHOTOMETER (KINGSBURY TYPE).

The arrangement of apparatus was similar in every respect to that with the Lummer-Brodhun. So far as possible, all noise and jar due to the rotating parts of the flicker attachment were eliminated, a precaution much more important in the case of the flicker photometer than in the case of the Lummer-Brodhun.

Within certain limits, the speed of the flicker photometer (light cycles per second) had little effect on the precision, as is shown by the following table:

TABLE V.

Showing Precision as a Function of Flicker Speed.

Effective illumination on photometer field approximately 25 meter-candles.
Observers: C, R, and T.
Color difference [12]: 4 wpc. carbon against 4 wpc. carbon filtered through the yellow (potassium bichromate) solution.

Speed (light cycles per second)	Total number of sets examined	Average a.d. for one setting	Average maximum deviation
		Per cent.	*Per cent.*
18	5	0.99	2.6
15	5	.95	2.1
12	5	.91	2.25
9	5	.94	2.5
6	5	1.32	3.75

Doubling the speed (*i.e.*, from 9 to 18 cycles per second) seemed to have little effect on the precision. In the data hereafter given a constant speed of 12 cycles per second was used.

[12] For brevity of expression we will hereafter designate this color difference as "carbon-yellow," and similarly the expressions "carbon-blue," "carbon-tungsten," "carbon-gas-filled," etc., will have an obvious meaning.

In the course of the experiments, a large number of preliminary measurements were made on the transmissions of the yellow and the blue solutions by several different observers, only one or two of whom had had even a brief experience with the flicker photometer. The following table therefore gives the precision attained by observers reading the flicker photometer for the first time:

TABLE VI.

Flicker Photometer.

Color difference	Observer	Total number of sets examined	Average a.d. for one setting	Average maximum deviation
			Per cent.	*Per cent.*
Carbon-yellow	B	22	1.12	2.88
Carbon-yellow	C	38	1.19	2.78
Carbon-yellow	F	8	1.22	2.87
Carbon-yellow	K	8	1.45	4.13
Carbon-yellow	M	7	.70	1.89
Carbon-yellow	N	9	.72	1.64
Carbon-yellow	R	36	.54	1.54
Carbon-yellow	S	8	1.48	3.84
Carbon-yellow	T	4	.96	2.21
Carbon-yellow	V	9	1.32	3.40
		Mean	1.07	2.71
Carbon-blue	B	22	0.88	2.27
Carbon-blue	C	28	.73	1.74
Carbon-blue	F	8	.88	2.27
Carbon-blue	K	8	1.06	2.86
Carbon-blue	M	7	.74	1.66
Carbon-blue	N	9	.82	2.16
Carbon-blue	R	31	.47	1.09
Carbon-blue	S	8	1.10	3.04
Carbon-blue	T	4	.94	2.51
Carbon-blue	V	9	1.01	2.23
		Mean	.86	2.18

It is thus seen that even with these color differences, on which measurements with the Lummer-Brodhun photometer were practically impossible, a group of 10 inexperienced observers maintained, in over 3000 individual settings, an average deviation of a single observation from the mean of its set of less than one per cent. At this rate, the precision of the mean of a set of 25 readings, expressed in terms of average deviation, would be about 0.2 per cent.—a figure which amply justifies the statement that the flicker photometer may be called a precision instrument.

It must not be inferred, however, from the above that a group of experienced observers would attain proportionately higher precision. The ability to make precise measurements with the flicker photometer seems to depend on the physiological peculiarities of the observer, and perhaps on his physical and mental condition, even more than in the case of the Lummer-Brodhun. One observer, C, improved markedly in precision as a result of several weeks' experience, while the opposite is true with another observer, R. It is difficult to say to what extent the precision attained depends upon practice.

COMPARISON OF FLICKER WITH EQUALITY-OF-BRIGHTNESS MEASUREMENTS.

With the eye-piece focused on the four-part field of the comparison prism, the flicker attachment, at rest, could be so turned as to bring both fields into view simultaneously. In this position equality-of-brightness measurements by the ordinary method were possible. The photometric field in this case was a circle of about 2° aperture, whose vertical diameter formed the dividing line between the two parts of the field. Using this device a number of settings were made with a color match, and with a carbon-tungsten color difference. These same measurements were repeated with the flicker method. Table VII gives the comparative precision.

TABLE VII.

Comparison of Flicker with Equality-of-Brightness Measurements in 2° Field.

Observers: B, C, F, K, M, N, S, R, V. Two sets each.

Photometer	Color difference	Number of sets examined	Average a.d. for one setting	Average maximum deviation
			Per cent.	*Per cent.*
Equality of brightness.....	None: Carbon-carbon	18	0.82	1.91
Flicker.....	None: Carbon-carbon	18	.87	1.86
Equality of brightness.....	Carbon-tungsten.....	18	1.74	4.07
Flicker.....	Carbon-tungsten.....	18	.99	2.48

Of course, one would not expect this type of equality-of-brightness photometer to give as precise results as the usual Lummer-Brodhun. It is, however, worth noting that while it and the flicker give, for the same group of observers, approximately the same precision with a color match, the loss of precision

with the carbon-tungsten color difference is small with the flicker, and much larger with the equality of brightness.

PRECISION OF THE FLICKER PHOTOMETER FOR COLOR DIFFERENCES COMMONLY MET IN PRACTICE.

To study this question advantage was taken of the flicker measurements with carbon-tungsten and with carbon-gas-filled color differences made simultaneously by the same group of observers which made the corresponding measurements with the Lummer-Brodhun contrast. (See Table IV.) In addition, measurements on the yellow and the blue solution, made at the same time, are appended for the sake of comparison. As before, the observers are divided rather arbitrarily into a more experienced group, Group A, and a less experienced group, Group P. The data is summarized in Table VIII.

TABLE VIII.

Photometer: Flicker.

Group A: Observers C, R, and T (more experienced).
Group P: Observers B, F, G, J, K, W (less experienced).

Color difference	Observers	Total number of sets examined	Average a.d. for one setting	Average maximum deviation
			Per cent.	*Per cent.*
None: carbon-carbon	Group A	18	0.68	1.55
	Group P	42	.87	2.09
Carbon-tungsten.....	Group A	6	.87	1.85
	Group P	14	.90	2.30
Carbon-gas-filled.....	Group A	12	.68	1.85
	Group P	28	1.13	2.75
Carbon-yellow.....	Group A	18	.85	2.05
	Group P	42	1.34	3.15
Carbon-blue.....	Group A	18	.75	1.80
	Group P	42	1.03	2.60

A comparison between Table VIII and Table IV shows that while with the carbon-tungsten color difference, the results obtained by the more experienced observers have a higher precision in the case of the Lummer-Brodhun than in the case of the flicker photometer, with the carbon-gas-filled color difference the advantage is distinctly in favor of the flicker. Further, the order of precision with the flicker photometer is practically the same for all color differences, and averages about 0.75 per cent. (average deviation of a single observation from the mean of its set). Expressed in terms of average deviation, the precision of

the mean of a set of 10 readings, taken by the better observers, would be about 0.25 per cent. In the observations with the regular Lummer-Brodhun photometer on the carbon-gas-filled color difference, the average deviation of a single observation from the mean (Table IV), by experienced observers, is 1.00 per cent. and the precision of the mean of a set of 10 readings should be about 0.30 per cent.

The agreement between the means of two or more sets is, however, a better indication of the precision of the measurement than can be obtained from a single set. As indicated in the preceding paragraph, the order of magnitude of the precision of the mean, for carbon-gas-filled color difference is 0.25 per cent. for the flicker photometer, and 0.30 per cent. for the Lummer-Brodhun, as computed from the observations of a single set. It is of interest to ascertain to what extent these values agree with actual measurements.

The data from which Table IV and Table VIII were computed were used for this purpose. In taking these data, each observer made two independent sets of observations for the several color differences, with each photometer. A short time-interval, during which other observations were made, always elapsed between the two sets. The deviation of the two means from their average value was computed, and taken as representing the order of magnitude of the uncertainty of the measurement. Values so obtained were averaged for the ten observers, as shown in Table IX.

TABLE IX.

Agreement Between the Means of Successive Sets of Readings.

Observers: Groups A and P combined (ten in all).

Photometer	Color difference	Average deviation of the mean	
		Observed	Computed*
		Per cent.	*Per cent.*
L–B contrast	None: Carbon-carbon	0.16	0.13
Flicker	None: Carbon-carbon	.19	.22
L–B contrast	Carbon-gas-filled	.90	.43
Flicker	Carbon-gas-filled	.27	.26

* This "computed" value was determined as follows: The average a.d. was computed for each case from the data given in Tables IV and VIII. This average a.d. was then divided by the square root of the average number of readings per set, which, while varying somewhat, was very nearly 15.

The approximate agreement between the observed and the computed values in both flicker measurements and in the contrast

photometer with no color difference is too consistent to be accidental. It shows that these measurements are reproducible to a reasonable degree. With the contrast photometer measurements on the carbon-gas-filled color difference the discrepancy between the observed and the computed values is very marked, and demonstrates conclusively what is a matter of common observation, that the individual observations in a single set of heterochromatic measurements taken with the ordinary equality-of-brightness or contrast photometer may agree fairly well among themselves, and the observations of a subsequent set may be in similar agreement, while the two sets may be so far apart that no two observations overlap. In other words, the measurements are not reproducible within the limits to be expected from examining the observations of a single set. It is as if the observer on making his first observation of a set, in his inability to decide when the differently colored fields are of the same brightness, makes a more or less arbitrary judgment, which for his eye, is probably correct within certain rather wide limits. His memory then helps him to make the next and succeeding observations in the same set with a reasonable precision, if the color difference is not too great. If some time elapses between the first and the second set of observations, his memory fails and he has to make a new, and probably a different, decision regarding the equality of brightness, which decision is followed rather consistently throughout that set. If he has practice enough his memory may hold over from one set to the next, and with more practice he may be able to settle upon a criterion which he follows thereafter, and which will differ in general from a similar criterion established by another observer, not so much because of any difference in the color vision of the two, as because of the accidental difference in starting point. If an observer, who had not already established his own criterion, were to make a large number of sets of observations on a given color difference with sufficient time between sets to wipe out the memory of the previous set, it is probable that this difference in accidental starting point could be averaged out, so that the result would give a true index of the observer's color sense, an index which, as shown in the paper by the writers to which reference has already been made, is given by the flicker photometer in a *single* set of observations.

CONCLUSIONS.

It has been shown that when using the Lummer-Brodhun contrast photometer for homochromatic measurements, experienced observers attained a precision, expressed in terms of average deviation of a single observation from the mean of its set, of approximately 0.3 per cent.; but that for heterochromatic measurements the precision decreased rapidly with increasing color difference. Further, except in the case of some observers of long practice, the heterochromatic measurements were not reproducible, the deviations of the means of successive sets being much greater than would be expected from the average deviation of a single observation from the mean of its set.

The precision attainable by the more experienced observers with the flicker photometer was in the neighborhood of 0.75 per cent. for the average deviation of a single observation. This value was not dependent on the color difference used, but was approximately the same for all color differences. Approximately, the ratio of the precision of the flicker to the Lummer-Brodhun photometer expressed in terms of deviation of a single observation, was for a color match 0.5; for the carbon-tungsten color difference, 1.0; and for the carbon-gas-filled, 1.5. Taking into account the greater reproducibility of the mean, it was found that even for the carbon-tungsten color difference the flicker photometer is the more precise instrument. The rather high field brightness used tended to favor the flicker photometer in these comparisons.

As stated at the beginning, the observations analyzed in this paper were not planned as a study of precision of measurement, and the data are by no means complete enough to indicate how the precision can best be increased. It is also realized that precision is only one of many factors which must be considered in choosing a photometer for any particular purpose, but it is a factor to which more study might profitably be given. In particular it should not be assumed that the sensibility or the precision is a constant characteristic of a type of instrument, but due recognition should be given to the physiological character of the ultimate instrument. Further increases in precision appear most likely to result from study of the conditions under which the eye operates best rather than from improvements in instruments.

PRELIMINARY NOTE ON THE RELATIONS BETWEEN THE QUALITY OF COLOR AND THE SPECTRAL DISTRIBUTION OF LIGHT IN THE STIMULUS.*

BY

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Bureau of Standards.

I. INTRODUCTION.

IT has long been established practice and is now recognized as a procedure of unquestioned validity to indicate the relative *brilliance* (brightness) of colors due to heterogeneous stimuli by the ratio of the integrals of light (" luminosity " or " luminous flux ") with respect to wave-length. It is also recognized that the *quality* of a color is determined by the spectral distribution of light, while, on the other hand, many different spectral distributions may excite colors of the same quality.[1]

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[1] Color is the *sensation* due to stimulation of the optic nerve.

The " adequate stimulus " of color is *light*.

Light (luminosity, luminous flux) is radiant power multiplied by the *visibility* of the radiant power in question. (*Radiant power* is the time rate of transfer of radiant energy.)

Relative visibility (function of wave-length) =

$$V_\lambda = \frac{P_{V\,max}}{P_\lambda}$$

where $P_{V\,max}$ is radiant power of the wave-length which requires a minimum of power to excite a given brilliance (brightness); P_λ is radiant power of wave-length, λ; and $P_{V\,max}$ and P_λ excite colors of equal brilliance.

By introspective analysis, color is found to possess three characteristics, the specification of which completely describes the color. These are: *hue, saturation, brilliance* (brightness). Hue and saturation are determined by the spectral distribution of the stimulus. It is convenient to designate as *quality*, the property of color determined by hue and saturation together. *Brilliance* is determined by the amount of the stimulus. In so far as it is permissible to ascribe quantitative significance to it, brilliance may be regarded as proportional to the logarithm of the stimulus. (Fechner's Law.) (For further discussion of definitions, and bibliography of same, see Priest, Op. Soc. of Am., Com. on Standards and Nomenclature, Subcom. on Colorimetry, Report, 1919 (Preliminary Draft), Bureau of Standards Library, Washington.)

The discussion in this present paper assumes brightness levels above the Purkinje effect.

But it does not appear that the general relations between colors due to various heterogeneous stimuli and the spectral distributions of these stimuli have ever been definitely formulated.[2] Indeed, it is not probable that any one has ever had available a collection of data sufficient to deal with this question as it is proposed to treat it in this paper. The author has now accumulated a considerable amount of data of diverse kinds which makes it possible to formulate one general condition which is always satisfied by two different spectral distributions which excite the same color. The purpose of this paper is to state this condition, together with some of the evidence supporting it.

II. DESCRIPTION OF ORIGINAL DATA USED AND THE GENERAL RELATION FOUND.

The primary data consist of pairs or sets of spectral distributions of radiant power which have been found by direct observation to excite colors of the same or nearly the same quality, although the distributions themselves differ markedly. These data may be classified as follows:

1. Data on Bittinger's Camouflage Paints.

There are six pairs of colored paints. The colors excited by the sunlight, reflected by the two members of each pair, are perfectly matched, or very nearly so, notwithstanding the fact that the spectral reflections of the two matched members are very different[3] (Figs. 1 and 2). The colors include two greens, two blues, a reddish purple and a brown.

---

[2] The manner of treatment followed in this paper is perhaps vaguely foreshadowed by the work of Sir Isaac Newton (*Optics*, London, 1730, 4th Ed., pp. 134 to 141), and H. Grassman (*Zur Theorie der Farbenmischung*, Pogg. Ann. 89, pp. 69–84; 1853: *Ges. Werke*, Leipzig, 1904, Vol. 2, part 2, pp. 161–173). It is obvious, however, on consideration of the nature of the data dealt with in the present paper, that the similarity of treatment can be only of the most general nature, for such quantitative data were necessarily entirely lacking to these early investigators. In modern works the closest approaches which the author has found to the problem are given by Parsons, "Introduction to the Study of Color Vision," Cambridge, 1915, Part i, Sec. ii, Chap. iii; H. E. Ives, "The Transformation of Color Mixture Equations," *Jour. Frank. Inst.*, Dec. 1915, pp. 673–701; and Luckiesh, "The Physical Basis of Color Technology," *Jour. Frank. Inst.*, July and August, 1917.

[3] Report to Chas. Bittinger on B. S. Test No. 28,479, June, 1920.

FIG. 1.

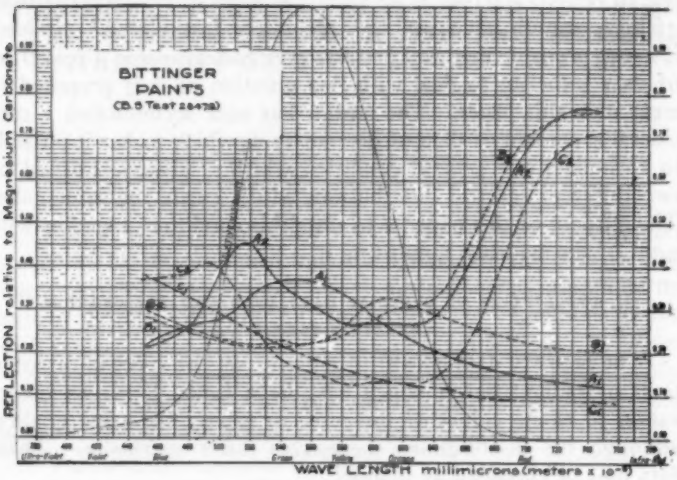
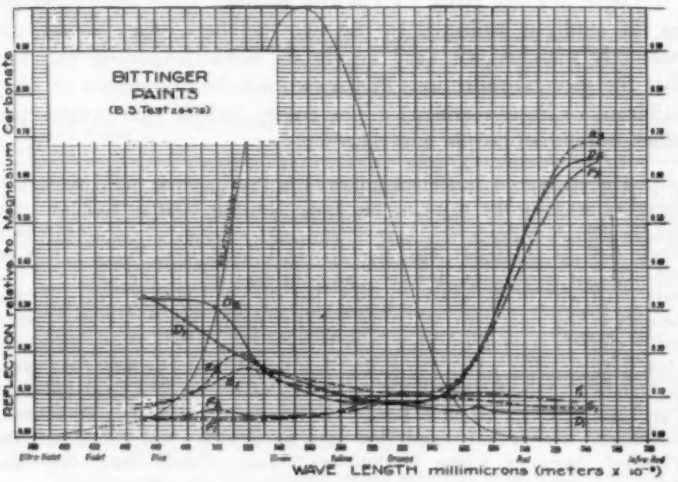

BITTINGER
PAINTS
(B. S. Test 26478)
REFLECTION relative to Magnesium Carbonate
WAVE LENGTH millimicrons (meters x 10⁻⁹)

FIG. 2.


BITTINGER
PAINTS
(B. S. Test 26479)
REFLECTION relative to Magnesium Carbonate
WAVE LENGTH millimicrons (meters x 10⁻⁹)

2. Data on Leucoscope[4] Color Matches.

There are two sets of data, each set comprising three spectral distributions. In each set the color due to the radiant power of a black body at a given temperature is approximately matched by the colors due, respectively, to two radiant powers of a markedly banded distribution, there being several bands with sharp maxima and minima distributed through the spectrum. The minima are actually zero; and the minima in one distribution coincide in wave-length with the maxima in the other distribution which excites the same or nearly the same color. The temperatures mentioned above are 2000° and 3000° K. (Figs. 9 and 10).

3. Data on Yellow Glasses, Yellow Oils and Color Matches of the Same Obtained by Rotatory Dispersion on the Arons Chromoscope.[5]

All of these stimuli are characterized by more or less absence of energy of short wave-lengths (blue and green), while the oils generally have marked absorption bands at about wave-lengths 610 and 670 millimicrons which do not occur in the glasses nor in the chromoscope[6] (Figs. 11 and 12). The colors range from about 35 yellow + 1 red to 35 yellow + 20 red on the Lovibond color scale. (Only part of the data available are published in this paper.)

These data have been treated as follows:

1. In the case of the data under 1 and 3, the observed or computed spectral reflection or transmission has been multiplied, by relative visibility and the relative energy of the sun, wave-

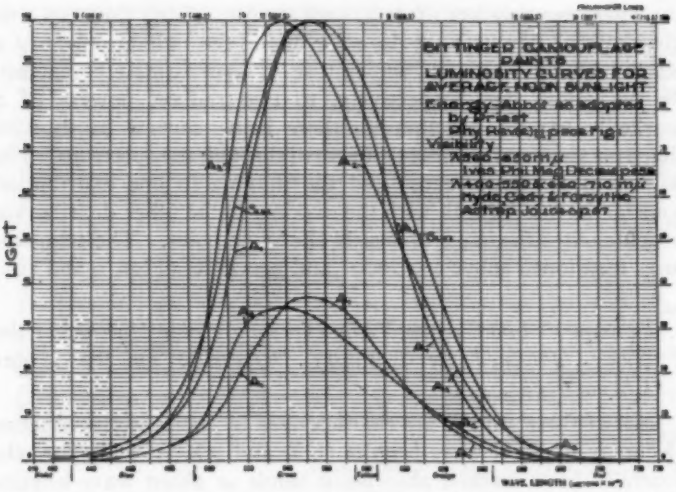
[4] Diro Kitao, Zur Farbenlehre, Inaug. Dis., Goettingen, 1878, and Abh. Tokio University, No. 12, 1885; Koenig, *Ann. der Phy.*, 17, pp. 990–1008, 1882; Brodhun, *Ann. der Phy.*, 34, pp. 897–918, 1888; Priest, "A New Study of the Leucoscope," forthcoming paper in JOUR. OF OPTICAL SOC. OF AMERICA.

[5] Leo Arons, *Ann. der Phy.* (4), 39, pp. 545–568, 1912; Priest, *Pro. Soc. Col. Prod. Analysts*, May, 1914. (There are several serious misprints.)

Priest, "The Application of Rotatory Dispersion to Colorimetry," *Phy. Rev.* (2), 15, pp. 538, 539, 1920; and forthcoming paper, JOUR. OPTICAL SOC. OF AMERICA.

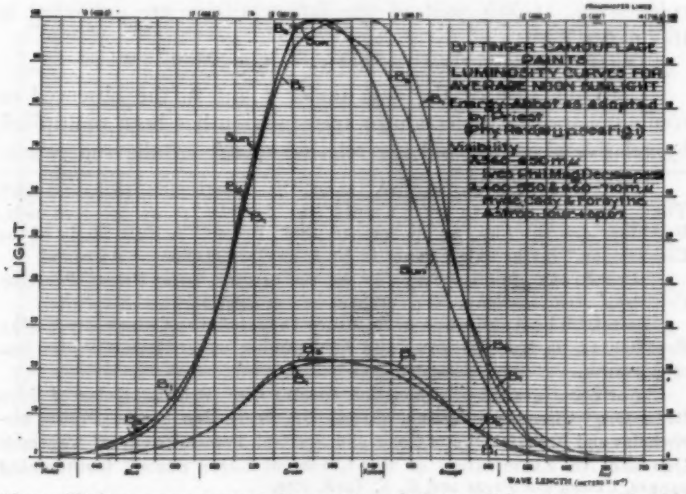
[6] For other graphs of absorption spectra see: Priest, "Color of Soya Bean Oil," Chemists' Section, *Cotton Oil Press*, Jan., 1920; Priest, McNicholas and Frehafer, "The Color and Spectral Transmissivity of Vegetable Oils and An Examination of the Lovibond Color Scale," forthcoming papers, *Cotton Oil Press* and *B. S. Tech. Pap.*

FIG. 3.



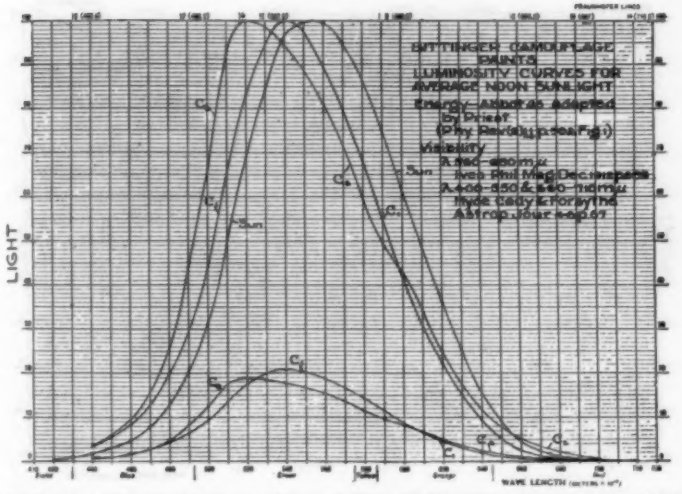
NOTE.—The lower curves represent the product:—*reflection* × *sunlight* × *visibility*. The higher curves are this same product *reduced to 100 at the maximum*.

FIG. 4.



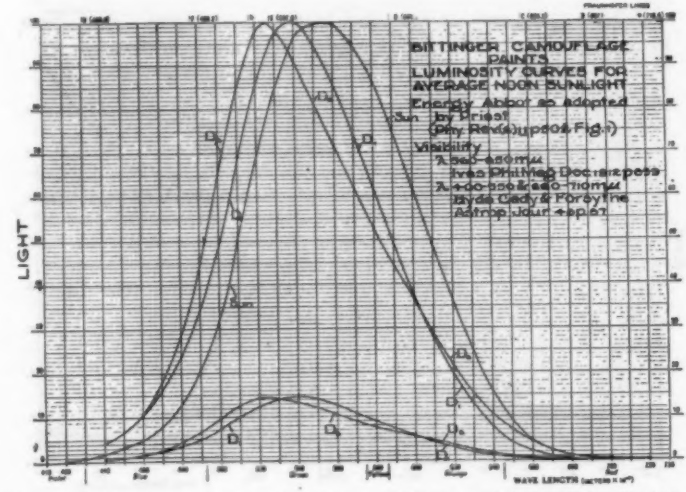
NOTE.—The lower curves represent the product:—*reflection* × *sunlight* × *visibility*,. The higher curves are this same product *reduced to 100 at the maximum*,

FIG. 5.



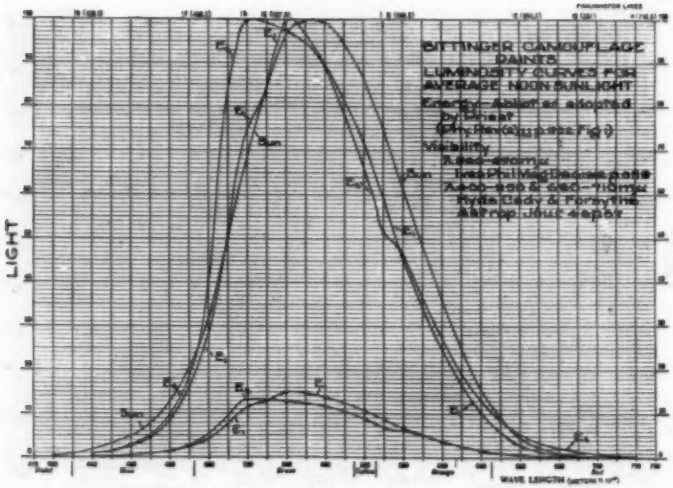
NOTE.—The lower curves represent the product:—*reflection* × *sunlight* × *visibility*. The higher curves are this same product *reduced to 100 at the maximum*.

FIG. 6.



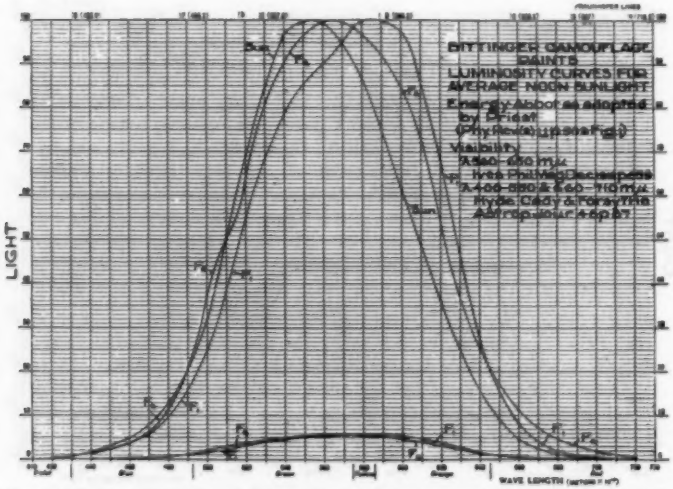
NOTE.—The lower curves represent the product:—*reflection* × *sunlight* × *visibility* The higher curves are this same product *reduced to 100 at the maximum*.

FIG. 7.


BITTINGER CAMOUFLAGE PAINTS
LUMINOSITY CURVES FOR AVERAGE NOON SUNLIGHT
Energy—Abbot as adopted by Priest (Phy. Rev. (1) p. 252 Fig. 1)
Visibility
λ 440–660 mμ
Ives Phil. Mag. Dec. 1912 p. 748
λ 400–430 & 660–710 mμ
Hyde, Cady & Forsythe
Astrophysical Journal 46 p. 137
LIGHT
WAVE LENGTH

NOTE.—The lower curves represent the product:—*reflection* × *sunlight* × *visibility*. The higher curves are this same product *reduced to 100 at the maximum*.

FIG. 8.


BITTINGER CAMOUFLAGE PAINTS
LUMINOSITY CURVES FOR AVERAGE NOON SUNLIGHT
Energy—Abbot as adopted by Priest (Phy. Rev. (1) p. 252 Fig. 1)
Visibility
λ 440–660 mμ
Ives Phil. Mag. Dec. 1912 p. 748
λ 400–430 & 660–710 mμ
Hyde, Cady & Forsythe
Astrophysical Journal 46 p. 137
LIGHT
WAVE LENGTH

NOTE—The lower curves represent the product:—*reflection* × *sunlight* × *visibility*. The higher curves are this same product *reduced to 100 at the maximum*.

FIG. 9.

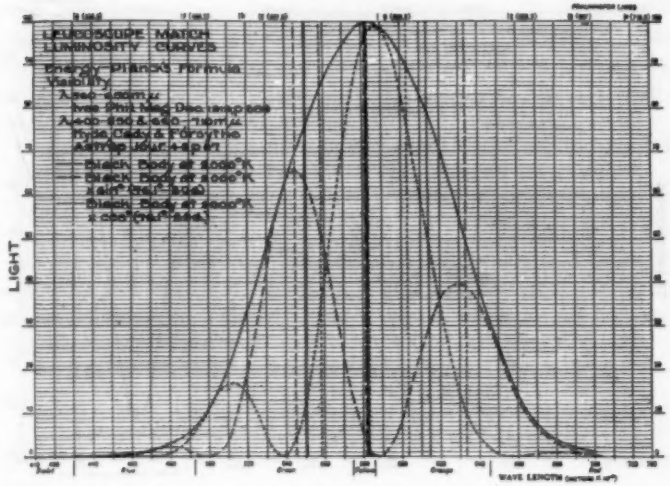

LEUCOSCOPE MATCH
LUMINOSITY CURVES
Energy-Planck's formula
Visibility
λ540-650mμ
Ives, Phil. Mag. Dec. 1912, p. 808
λ440-490 & 490-710mμ
Hyde, Cady & Forsythe
Astrophysical Journal
Black Body at 4000°K
Black Body at 4000°K
Black Body at 4000°K
LIGHT
WAVE LENGTH

FIG. 10.

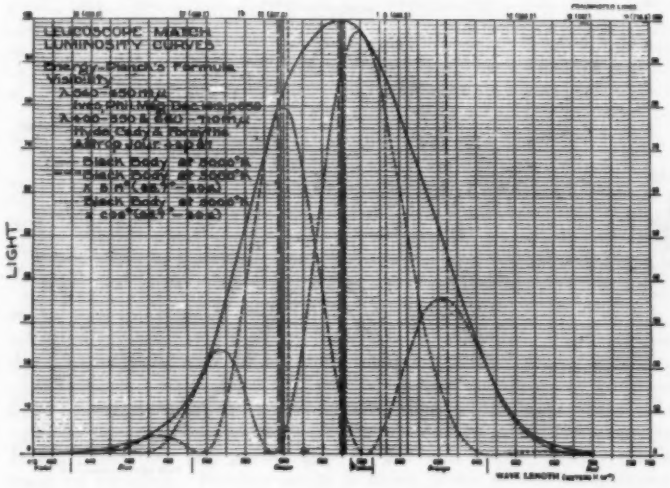

LEUCOSCOPE MATCH
LUMINOSITY CURVES
Energy-Planck's Formula
Visibility
λ540-650mμ
Ives, Phil. Mag. Dec. 1912, p. 808
λ440-490 & 490-710mμ
Hyde, Cady & Forsythe
Astrophysical Journal
Black Body at 4000°K
Black Body at 4000°K
Black Body at 4000°K
LIGHT
WAVE LENGTH

FIG. 11.

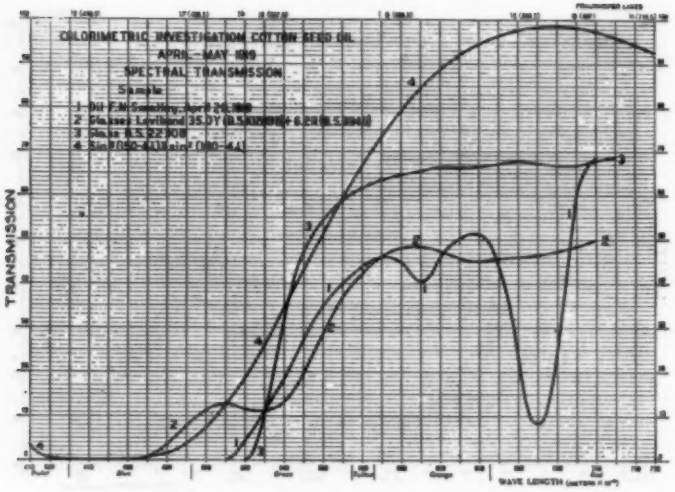
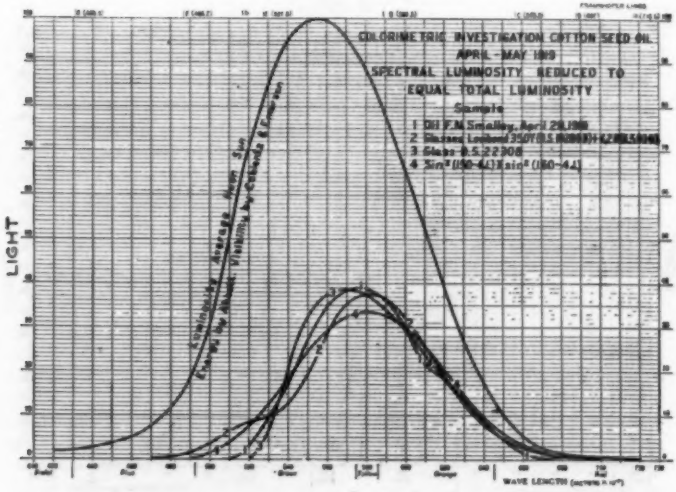

COLORIMETRIC INVESTIGATION COTTON SEED OIL
APRIL - MAY 1919
SPECTRAL TRANSMISSION
Samples
1 Oil F.N. Smalley, April 29, 1919
2 Glasses Lovibond 35.0Y (0.5 R+0.5 B)+0.2R (0.5 R+0.5 B)
3 Glass B.S. 22308
4 Sin²(150-4λ) sin² (180-4λ)
TRANSMISSION
WAVE LENGTH

FIG. 12.


COLORIMETRIC INVESTIGATION COTTON SEED OIL
APRIL - MAY 1919
SPECTRAL LUMINOSITY REDUCED TO
EQUAL TOTAL LUMINOSITY
Samples
1 Oil F.N. Smalley, April 29, 1919
2 Glasses Lovibond 35.0Y (0.5 R+0.5 B)+0.2R (0.5 R+0.5 B)
3 Glass B.S. 22308
4 Sin²(150-4λ) sin² (180-4λ)
LIGHT
WAVE LENGTH

length by wave-length. In the case of the leucoscope data (2), relative energy by Planck's formula has been multiplied by visibility and the rotatory dispersion transmission. In all cases the resulting function of wave-length is *light* or "luminosity"; and the graph with this function as ordinate and wave-length as abscissa is a so-called "luminosity curve." For this purpose I have used data as follows:

(*a*) Spectral distribution of the sun's energy from Abbot as adopted by Priest.[7]

(*b*) Visibility: For wave-lengths 560 to 650 millimicrons, by Ives[8]; for wave-lengths 400 to 550 and 660 to 700, by Hyde, Forsythe and Cady.[9] (The author's own visibility, as determined by Coblentz and Emerson, is in close agreement with this average data.) In Table III and Fig. 12, visibility by Coblentz and Emerson[10] has been used.

2. The "luminosity curves" mentioned above have been plotted.

3. The wave-length of the centre of gravity of each of these curves has been determined.[11] This wave-length is designated as λ_c.

Considering all of these data (Tables I, II, III), the following rule has been found to hold without exception:

If any two lights, however different in spectral distribution, excite colors of the same quality, the wave-lengths of the centres of gravity of their spectral distributions are coincident.[12] The

[7] *Phy. Rev.* (2), 11, p. 502, 1918, Fig. 1, open circles.

[8] *Phil. Mag.*, Dec., 1912, p. 859.

[9] *Astrop. Jour.*, 48, p. 87, 1918.

[10] *B. S. Sci. Pap.* 303, Table 5.

[11] By means of an Amsler planimeter, making several check determinations. In nearly all cases, check determinations have been made by different operators in order to avoid blunders. In a few cases, check determinations have also been made by mechanical balancing on a knife edge. The final results obtained are accurate to about 0.5 millimicron.

[12] The generality of this conclusion is, of course, limited by the extent of the data considered. The reader will note, however, the diversity of the data, including the confirmatory data on monochromatic analysis in the latter part of this paper. The converse of this proposition is not necessarily true; two spectral distributions of light may have the same wave-length of centre of gravity and *not* excite colors of the same quality if the lights in the two cases are distributed over different ranges of wave-length. In order that they may excite colors of the same quality another condition must be satisfied. Further study is being given to the formulation of this condition.

TABLE I.

Billinger's Paints.

Paint ↓	λ → / Color ↓	Reflection relative to $MgCO_3$ 460	500	520	540	560	580	600	620	640	660	700	740	Total sunlight	λ_c
A_1	Green*	0.23	0.28	0.33	0.37	0.36	0.34	0.28	0.24	0.21	0.18	0.15	0.13	0.320	558.2
A_2		0.22	0.38	0.45	0.36	0.31	0.27	0.27	0.27	0.28	0.39	0.67	0.77	0.324	555.8
B_1	Reddish	0.27	0.22	0.21	0.21	0.22	0.26	0.32	0.32	0.29	0.27	0.23	0.21	0.248	565.4
B_2	purple	0.28	0.23	0.22	0.23	0.22	0.24	0.28	0.31	0.33	0.43	0.70	0.76	0.246	564.4
C_1	Blue	0.36	0.29	0.25	0.22	0.19	0.16	0.14	0.13	0.12	0.10	0.09	0.09	0.196	548.8
C_2		0.37	0.30	0.28	0.18	0.15	0.13	0.13	0.13	0.14	0.22	0.55	0.71	0.191	547.1
D_1	Blue	0.32	0.22	0.18	0.15	0.12	0.10	0.09	0.08	0.07	0.07	0.06	0.06	0.135	546.1
D_2		0.32	0.30	0.21	0.13	0.10	0.09	0.09	0.09	0.09	0.15	0.47	0.65	0.139	544.4
E_1	Green	0.08	0.13	0.16	0.16	0.14	0.12	0.11	0.10	0.10	0.09	0.08	0.07	0.132	555.9
E_2		0.08	0.17	0.19	0.13	0.12	0.10	0.10	0.10	0.11	0.15	0.47	0.69	0.125	554.2
F_1	Reddish	0.04	0.04	0.05	0.05	0.05	0.07	0.09	0.11	0.10	0.11	0.10	0.09	0.0633	571.8
F_2	brown	0.05	0.07	0.05	0.05	0.06	0.06	0.08	0.08	0.09	0.14	0.42	0.63	0.0646	568.2

* A_1 is a slightly more yellowish green than A_2, which corresponds with their respective values of λ_c. The other pairs (paints designated by the same letter) are very closely color-matched. Compare with Figs. 1 to 8, inclusive.

TABLE II.

Leucoscope.

Fig. No.	Curve	λ_c	Relative color by direct observation
9	Dashed	581.5	Pale bluish.
	Solid	582.6	
	Dotted	583.2	Yellowish, more saturated.
10	Dashed	568.9	Bluish.
	Solid	570.8	
	Dotted	571.9	Yellowish.

NOTE: Compare with spectral light distributions in Figs. 9 and 10. The three distributions shown in each figure excite colors of the same or nearly the same quality. The slight relative differences in quality are indicated in the last column of this table, and it will be observed that they correspond with the small differences in λ_c.

TABLE III.

Oil, Glasses and Rotatory Dispersion (Arons Chromoscope).

Numbers of curves, Figs. 11 and 12	λ_c
1	583.2
2	581.8
3	583.3
4	583.4

NOTE: Compare with spectral transmission curves in Fig. 11 and luminosity curves in Fig. 12. These four samples excite colors of the same or nearly the same quality in daylight.

small discrepancies are of the same order of magnitude as the wave-length hue limen, that is, about 1 millimicron (metre $\times 10^{-9}$).[13] In the case of colors, nearly, but not perfectly matched in quality, the small differences observed in λ_c are in the sense to be expected from the apparent slight difference in hue. (A_1 and A_2, Table I; and Table II.)

III. RELATION TO MONOCHROMATIC ANALYSIS AND CONFIRMATION BY PREVIOUS DATA.

"Monochromatic analysis"[14] may be regarded as a special case of finding, by trial, a particular type of spectral distribution which will excite color of the same quality as another distribution.[15] In the mixture of homogeneous light and sunlight determined by this method, the wave length of the centre of gravity will be given by:

$$\lambda_c = \frac{\Lambda l_\Lambda + \lambda_{cs} l_s}{l_\Lambda + l_s} \qquad (1)$$

where

Λ = wave length of dominant hue,
λ_{cs} = wave length of centre of gravity for sunlight,
= 560.0 millimicrons,

l_Λ and l_s are respectively the amounts of homogeneous light and sunlight mixed.

If $\Pi = \frac{l_\Lambda}{l_\Lambda + l_s}$, this reduces to:

$$\lambda_c = \Pi\Lambda + \lambda_{cs}(1 - \Pi). \qquad (2)$$

Λ and Π being the data given by monochromatic analysis, it is obvious that λ_c for the mixture of homogeneous light and sunlight can be readily computed by this equation. If the rule stated above[16] is universally valid, this value of λ_c should equal the value of λ_c for any spectral distribution which excites color

[13] Steindler, *Wien. Sitz.* 115, 2 A., p. 39, 1906; Nutting, *Bull. B. S.* 6, pp. 89-93, 1909. (The decimal point in the tabulated values column 2, Table II, is one place too far to right.) Jones, JOUR. OPTICAL SOC. OF AMERICA, 1, pp. 63-77, 1917.

[14] Nutting, *Bull. B. S.* 9, p. 1, 1913.

[15] Op. Soc. of Am., Com. on Standards and Nomenclature, Sub-com. on Colorimetry, Report, 1919 (Preliminary Draft), pp. 28 and 38. Copy in Bur. Stands. Library, Washington.

[16] Page 397.

of the same quality as the color excited by the mixture of homogeneous light and sunlight specified by Λ and Π. I have tested this in a number of cases and have confirmed the rule in every instance.

The data used for this purpose may be classified as follows:

1. Jones' data on Λ and Π for a number of fairly reproducible light sources,[17] *together with Hyde and Forsythe's data on the color temperature of the same sources.*[18] We have computed λ_c from "color temperature" on the basis of Planck's formula and the same visibility as described above (Ives and Hyde, Forsythe and Cady); and also λ_c from Jones' values of Λ and Π. The results are compared in Table IV.

TABLE IV.

Source	From color temp.		From Monoc. analysis		
	Temp.	λ_c	Λ	Π	λ_c
Hefner	1875° K.	585.0	593.0	0.86	588.3
Pentane	1914	584.0	592.0	0.85	587.1
Tung, 1.25 w. p. c.	2385	577.2	588.0	0.65	578.0
Acetylene	2360	577.5	585.5	0.64	576.4

2. Jones' data on Λ and Π for Soya bean oil of color 35 yellow +7.2 red, Lovibond scale,[19] *together with our own average value of λ_c derived from the actual spectral distribution of sunlight transmitted by oil of this same color on the Lovibond scale.* The results are:

	λ_c
From Jones' $\Lambda = 587.7$ and $\Pi = 0.84$	583.3
From our distribution data	583.4

It appears on consideration of these results, and those in Table IV, that the rule in regard to equality of the values of λ_c for different spectral distributions which excite colors of the same quality is confirmed to within the reliability of the data. It is to be remembered that Λ and Π on one hand and "color temperature" on the other were not determined for the same sources, but for sources more or less accurately reproduced from specifications; and it will be noticed that the agreement of the

[17] L. A. Jones, *I. E. S.* **9**, p. 691, 1914.

[18] Hyde and Forsythe, *Jour. Frank. Inst.*, **183**, pp. 353, 354, 1917.

[19] Wesson, *Cotton Oil Press*, Chemists' Section, Table I, p. 66, July, 1920.

values of λ_c is closest for the source which is the most accurately reproducible, the vacuum tungsten lamp at 1.25 w.p.c.

The following corollaries from equation (2) present points of interest:

1. If $\Pi = 1$ or if $\Lambda = \lambda_{cs}, \lambda_c = \Lambda$.

2. If $\Pi \neq 1$ and $\Lambda \neq \lambda_{cs}$, Λ lies on the side of λ_c remote from λ_{cs}.

3. If Π = zero, Λ is indeterminate as would be expected (since, if Π = zero, λ_c must equal λ_{cs}).

The author is indebted to Paul D. Foote for advice on the method of determining the centre of gravity and to K. S. Gibson, H. J. McNicholas, Mabel K. Frehafer, C. M. Blackburn, M. C. Malamphy, C. L. Snow, J. H. Turner, Jr., and Gertrude Clemens for assistance in observations and in carrying through the intricate computations and plotting of curves necessary to obtain the above results.

NATIONAL BUREAU OF STANDARDS,
August 9, 1920.

NOTE ON THE RELATION BETWEEN THE FREQUENCIES OF COMPLEMENTARY HUES.*

BY

IRWIN G. PRIEST.

THERE are available considerable data on the wave-lengths of complementary hues.[1] It appears that such data have always been presented and discussed in terms of wave-length, and not in terms of frequency.[2] It has been suggested that the curves relating complementary wave-lengths "nearly resemble" (rectangular) "hyperbolæ."[3] Inspection of Fig. 1 will show that the approximation to a rectangular hyperbola is not at all close; the alleged likeness is rather far-fetched.

It might reasonably be expected that the relation between frequencies would be simpler than that between wave-lengths. From the point of view of physiologic optics, wave-length can be nothing more than a purely arbitrary reference scale; while it is at least a reasonable hypothesis that some comparatively simple relations may exist between the retinal response and the *frequency* of the stimulus. The author has previously shown that retinal visibility as a function of frequency is simpler than as a function of wave-length.[4] In this paper it is proposed to show that the relation between the frequencies of complementary hues can be represented in a simple way which is at least suggestive of simple relations between the retinal response and frequency.

The data mentioned above have been replotted, as shown in Fig. 1. A curve has been drawn to represent, as well as may be, these data. From this curve pairs of complementary wave-

* Published by permission of the Director, Bureau of Standards.

[1] Helmholtz, v. Kries, v. Frey, Koenig, Angier, Trendelenburg and Diterici as quoted in Helmholtz: "Physiol. Optics," 3rd Ed., Vol. 2, p. 107.

[2] Helmholtz: "Physiol. Optics," 2nd Ed. (1896), pp. 316–319. Helmholtz: "Physiol. Optics," 3rd Ed. Vol. 2, pp. 105–107. Parsons: "Introduction to Study of Color Vision" (Cambridge, 1915) Part I, Sec. II, Chap. III. Rood: "Modern Chromatics" (New York, 1879), p. 175. Köllner: "Störungen des Farbensinnes" (Berlin, 1912), p. 14. Luckiesh: "Color and Its Applications" (New York, 1915), pp. 59 and 75. Grünberg: Sitzb. Ak. Wein, II A, 113, pp. 627–636; 1904.

[3] Parsons: "Introduction to Study of Color Vision," p. 35. Grünberg: Sitzb. Ak. Wien, II A, 113, pp. 627–636; 1904.

[4] *Phy. Rev.* (2) 11, 498, 1918.

FIG. 1.

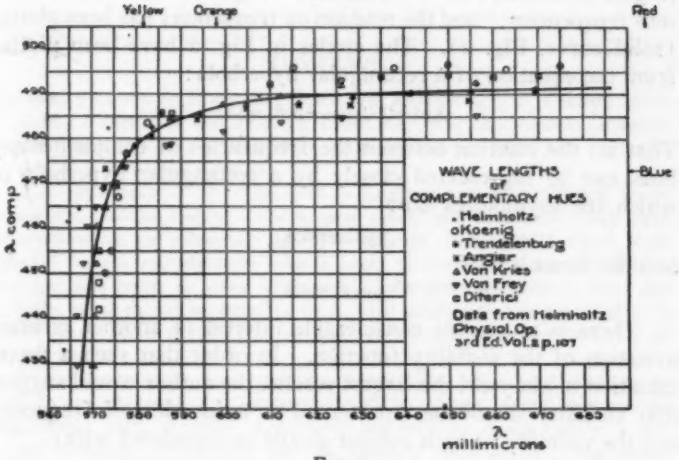
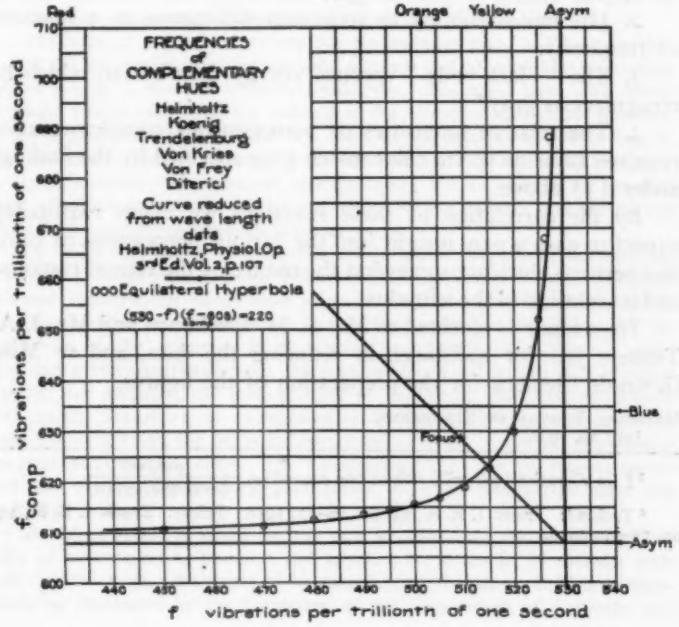

Yellow
Orange
Red
Blue
λ comp
λ
millimicrons
WAVE LENGTHS
of
COMPLEMENTARY HUES
. Helmholtz
o Koenig
+ Trendelenburg
x Angier
△ Von Kries
▼ Von Frey
o Diterici
Data from Helmholtz
Physiol. Op.
3rd Ed. Vol. 2 p. 107

FIG. 2.


Red
Orange
Yellow
Asym
Blue
Asym
Focus
f comp
f vibrations per trillionth of one second
vibrations per trillionth of one second
FREQUENCIES
of
COMPLEMENTARY
HUES
Helmholtz
Koenig
Trendelenburg
Von Kries
Von Frey
Diterici
Curve reduced
from wave length
data
Helmholtz, Physiol. Op.
3rd Ed. Vol. 2, p. 107
ooo Equilateral Hyperbola
(530 - f)(f - 508) = 220

lengths have been read. These wave-lengths have been converted into frequencies;[5] and the relation of frequencies has been plotted (solid curve, Fig. 2). The circles in Fig. 2 have been plotted from the equation of a rectangular hyperbola:

$$(530-f)\,(f_{comp}-608)=220$$

That is: the relation between the frequencies of complementary hues can be represented closely by a rectangular hyperbola of which the asymptotes are:

530 and 608

and the focus is:

509.2, 628.8

There is at present considerable interest in another redetermination of the visibility function. In order that such a determination might yield the largest results, the author would suggest that visibility should be represented as a function of frequency and the visibility of each subject should be correlated with:

1. The frequencies of his spectral complementaries.
2. His hue sensibility to frequency differences as a function of frequency.
3. The "black-body" spectral energy distribution which he recognizes as gray.[6]
4. The relative intensities of homogeneous complementaries required to be mixed to color match gray specified by the findings under (3) above.

By the correlation of these functions we might reasonably expect to gain a new insight into the various phenomena of color and perhaps elucidate somewhat the nature of the retinal response and its relation to the stimulus.

The author is indebted to Mr. C. M. Blackburn and Mr. J. A. Turner, Jr., for assistance in reducing the data, and to Miss Gertrude Clemens for the preparation of the figures.

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[5] f (in vibrations per trillionth of one second) $=\dfrac{300{,}000}{\lambda \text{ (in millimicrons)}}$

[6] Troland: Trans. I. E. S., 13, pp. 26–27; 1918. Priest: Trans. I. E. S., 13, pp. 75–77; 1918.

REVIEWS.

THE ORGANIZATION OF INDUSTRIAL SCIENTIFIC RESEARCH, C. E. Kenneth Mees, D.Sc., Director of the Research Laboratory, Eastman Kodak Company. 170 pages, 5½ × 8, illustrated with 10 charts; $2.00 net, postpaid; McGraw Hill Book Company.

The experience of Dr. Mees both in England and in the United States abundantly qualifies him to write a most timely and instructive book upon the organization of industrial, scientific research. His book is stated to be intended for the manufacturer who is ready to organize a research laboratory, but who wants to know what such a laboratory will cost to start and maintain, what should be expected of it, what position in the organization it should occupy, how it should be directed, and where competent men may be secured for it.

Following a most comprehensive introduction, there are chapters on research and coöperative laboratories, on their relation to the plant, on the organization, staff, direction, equipment, and building design; there is also a very complete bibliography and an index of laboratories. The manufacturer will find here concise and ready answers to perhaps all but the last question—How may competent research men be obtained?

The book, if apparently written for the business man, will be nevertheless of the greatest interest to scientific workers and all who are responsible for, or having to do with, scientific production. The pages teem with words of sound advice regarding the relation of the scientific man to his problem, to the organization of which he is a part, to his fellows, to publication, to the community. Much of what is so ably set forth is applicable not only to the laboratories of industrial establishments but to centres of research generally, and one cannot be but impressed by the orderly and systematic method of conducting research here set forth.

Dr. Mees considers very promising the future of coöperative laboratories, for types of industry with many relatively small or scattered units, especially in those industries in which little scientific work has been done and where no strong feeling of exclusiveness and secrecy prevails. He well states the limitations of the university in its relation to industrial research, but could have put greater emphasis upon the possibilities of the coöperative regional aspects; for example, the inter-relation of certain of the technical schools with nearby industries, as exemplified in the Sheffield Scientific School of Yale University and the brass industry; the Carnegie Technical School and the steel industry, and others. The "Technology Plan" of the Massachusetts Institute, of very far reaching possibilities, has been inaugurated since this book was written.

Dr. Mees seems to stand on less sure ground when discussing the relations of government to research, and appears, for example, to consider with some hesitancy the advantages to be gained from the recent British developments as illustrated by the formation of the Department of Scientific and

Industrial Research. Again, the rôle of the Government of the United States in the fostering of industrial research is hardly given in its proper perspective; thus the great, industrial, scientific, research organization which is the Department of Agriculture is passed over all too briefly. The part played both directly and indirectly by such government institutions as the Bureau of Standards in organizing, executing, and stimulating industrial research in its own and other laboratories, and its service to industry in furnishing both ideas and men accomplished in research, could have well received more attention.

Industrial laboratories are classified in three general divisions: (1) works laboratories exerting analytical control over materials, processes, and product; (2) industrial laboratories working on improvement in product and in process, tending to lessen cost of production and to introduce new products on the market; (3) laboratories working on pure theory and on the fundamental sciences associated with the industries; and their respective scope and relation one to the other are well set forth. In this connection an interesting phenomenon is to be noted in the evolution of industrial research laboratories that in certain striking instances they have been practically compelled not only to act as development laboratories and semi-manufacturing plants, but actually to take over the permanent manufacturing of special products originating in the laboratory; in other words, industry developing from the laboratory rather than an industry establishing a laboratory.

Laboratories may also be grouped as "divergent" or "convergent" depending upon whether the problems investigated are all connected with one subject or cover a disjointed or scattered range of subjects. For either type there are two forms of organization possible, designated as "departmental" and "cell" systems which are discussed in detail.

The chapters on the staff and direction of a research laboratory are perhaps of greatest interest to the scientific man; they are based largely on the highly satisfactory experience and practice of the author at the Eastman Kodak Plant, although he has drawn generously from the writings and experience of others.

It is encouraging to note that Dr. Mees frowns on secrecy, although maintaining this is necessary in very exceptional cases, and secret processes should be confined to very few people. The spirit of the laboratory should be always to encourage publication.

G. K. BURGESS,
August, 1920.

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